Smoothing Stochastic Bang-Bang Problems

with Application to the Optimal Execution Problem

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Dipl.-Math., Dipl.-Volksw. Katrin Eichmann, geb. Bernoth

Präsident der Humboldt-Universität zu Berlin:
Prof. Dr. Jan-Hendrik Olbertz

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät II:
Prof. Dr. Elmar Kulke

Gutachter:
1. Prof. Dr. Peter Imkeller
2. Prof. Dr. Ying Hu
3. Prof. Dr. Michael Kupper

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Abstract

Motivated by the problem of how to optimally execute a large stock position, this thesis considers a stochastic control problem with two special properties. First, the control problem has an exponential delay in the control variable, and so the present value of the state process depends on the moving average of past control decisions. Second, the coefficients are assumed to be linear in the control variable.

It is shown that a control problem with these properties generates a mathematically challenging problem. Specifically, it becomes a stochastic control problem whose solution (if one exists) has a bang-bang nature- i.e., the optimal solution switches discontinuously between a given upper and lower bound. This discontinuity creates difficulties in proving the existence of an optimal solution and in solving the problem with numerical methods.

A sequence of stochastic control problems with state processes is constructed, whose diffusion matrices are invertible and approximate the original degenerate diffusion matrix. The cost functionals of the sequence of control problems are convex approximations of the original linear cost functional. To prove the convergence of the solutions, the control problems are written in the form of forward-backward stochastic differential equations (FBSDEs). It is then shown that the solutions of the FBSDEs corresponding to the constructed sequence of control problems converge in law, at least along a subsequence. By assuming convexity of the coefficients, it is then possible to construct from this limit an admissible control process which, for an appropriate reference stochastic system, is optimal for our original stochastic control problem. In addition to proving the existence of an optimal (bang-bang) solution, another useful result is obtained: The optimal solutions of the sequence of approximating control problems exhibit a smooth approximation of the discontinuous optimal bang-bang solution and can be used for the numerical solution of the problem.

These results are then applied to the optimal execution problem to find an optimal strategy for executing a given stock position such that the expected execution price is maximized. The trading activity is assumed to have an impact on the stock price- either because the stock is very illiquid or because the size of the trade is large enough to affect the market price. This problem is formulated as a stochastic control problem where the control variable denotes the trading speed. The impact on the stock price is assumed to have a transient component which decays over time. This means that the choice of trading speed influences not only the current but also the future stock price. Arguments are presented concerning the assumption that the impact of the trading speed on the stock price is linear and the decay of the transient price impact is exponential. With these assumptions the solution of the mathematically challenging problem which was previously treated theoretically is obtained. An approximating control problem for the numerical solution of the kind derived earlier is chosen and hence a smooth approximation of the optimal bang-bang problem is achieved. Numerical results for the case of a fast decay of the transient impact are compared with the case of a slow decay for both constant and time-varying liquidity of the stock.

Keywords: Stochastic control, stochastic bang-bang problems, smooth approximation of bang-bang solutions, optimal execution problem.

Zusammenfassung

Motiviert durch das Problem der optimalen Strategie beim Handel einer großen Aktienposition, behandelt diese Arbeit ein stochastisches Kontrollproblem mit zwei besonderen Eigenschaften. Zum einen wird davon ausgegangen, dass das Kontrollproblem eine exponentielle Verzögerung in der Kontrollvariablen beinhaltet, das heißt, dass der gegenwärtige Wert des Zustandsprozesses von einem gleitenden Mittel der vergangenen Kontrollentscheidungen abhängt. Zum anderen nehmen wir an, dass die Koeffizienten des Kontrollproblems linear in der Kontrollvariablen sind.
Es wird gezeigt, dass ein Kontrollproblem dieser Art zu folgendem mathematisch herausfordernden Problem führt: Wir erhalten ein degeneriertes stochastisches Kontrollproblem, dessen optimale Lösung - sofern sie existiert - Bang-Bang-Charakter hat, das heißt, dass die optimale Lösung unstetig zwischen zwei vorgegebenen Extremwerten wechselt. Diese Unstetigkeit der optimalen Kontrolle führt dazu, dass die Existenz einer optimalen Lösung nicht selbstverständlich ist und bewiesen werden muss. Außerdem stellt die Unstetigkeit eine große Schwierigkeit für die numerische Lösung des Problems dar.
Es wird eine Folge von stochastischen Kontrollproblemen mit Zustandsprozessen konstruiert, deren jeweilige Diffusionsmatrix invertierbar ist und die ursprüngliche degenerierte Diffusionsmatrix approximiert. Außerdem stellen die Kostenfunktionale der Folge eine konvexe Approximation des ursprünglichen linearen Kostenfunktionals dar. Um die Konvergenz der Lösungen dieser Folge zu zeigen, stellen wir die Kontrollprobleme in Form von stochastischen Vorwärts-Rückwärts-Differentialgleichungen (FBSDEs) dar. Wir zeigen, dass die zu der konstruierten Folge von Kontrollproblemen gehörigen Lösungen der Vorwärts-Rückwärtsgleichungen - zumindest für eine Teilfolge - in Verteilung konvergieren. Mit Hilfe einer Konvexitätsannahme der Koeffizienten ist es dann möglich, einen Kontrollprozess auf einem passenden Wahrscheinlichkeitsraum zu konstruieren, welcher optimal für das ursprüngliche stochastische Kontrollproblem ist. Neben der damit bewiesenen Existenz einer optimalen (Bang-Bang-) Lösung, erhält man noch ein weiteres nützliches Resultat: Die optimalen Lösungen der Folge von approximierenden Kontrollproblemen stellen eine glatte Approximation der unstetigen optimalen Bang-Bang-Lösung dar, welche man für die numerische Approximation des Problems verwenden kann.
Die Ergebnisse werden dann auf das Problem der optimalen Handelsausführung angewendet. Dessen Ziel ist, eine optimale Strategie dafür zu finden, eine gegebene Aktienposition am Markt so zu verkaufen, dass man den höchsten erwarteten Verkaufspreis erzielt. Dabei geht man davon aus, dass die Aktienposition sehr groß oder die Aktie sehr illiquide ist, so dass die Handelsaktivität einen Einfluss auf den Aktienpreis hat. Das Problem wird als ein stochastisches Kontrollproblem mit der Handelsgeschwindigkeit als Kontrollvariable formuliert. Es wird angenommen, dass der Einfluss auf den Aktienpreis eine temporäre Komponente hat, welcher mit der Zeit ausklingt. Das bedeutet, dass die gegenwärtig gewählte Handelsgeschwindigkeit nicht nur den jetzigen, sondern auch den zukünftigen Preis beeinflusst. Wir liefern Argumente dafür, den Einfluss der Handelsgeschwindigkeit auf den Aktienpreis als linear und das Ausklingen des temporären Preiseinflusses als exponentiell anzunehmen. Damit befinden wir uns in der Situation des mathematisch herausfordernden Problems, welches wir zuvor theoretisch behandelt haben. Folglich wird für die numerische Lösung ein approximierendes Kontrollproblem in der Art, wie es zuvor

hergeleitet wurde, konstruiert, wodurch man eine glatte Approximation der optimalen Bang-Bang-Lösung erhält. Die numerischen Ergebnisse werden für den Fall eines schnellen Abklingens mit dem Fall des langsamen Abklingens des temporären Einflusses verglichen, wobei zunächst von einer konstanten Liquidität der Aktie ausgegangen und im Anschluss der Einfluss von zeitabhängiger Liquidität auf die optimale Lösung betrachtet wird.

Schlagwörter: Stochastische Kontrolltheorie, stochastische Bang-Bang Probleme, glatte Approximierung von Bang-Bang Lösungen, optimale Handelsausführung.

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1. Motivation - The Optimal Execution Problem

During the last years investments in equities grew rapidly and led to a rising concentration of assets among financial institutions. It is not unusual that a portfolio manager supervises a large portfolio with individual positions that might contain a significant fraction of the average daily volume of the security. Rebalancing such portfolios generates considerable orders across many stocks and must often be executed within a relatively short time horizon. Due to the illiquidity of the equity markets, orders of this size create significant impact on the asset price. Therefore, in order to reduce the price impact, it is necessary to split the large order into smaller pieces. But splitting the order into smaller pieces means that it takes a longer time until it is executed. During this time the market price can move significantly due to exogenous events. Thus, there is a tradeoff between trading cost and trading risk and the challenge is to find the optimal size and timing of the suborders such that the total expected costs due to the price impact is minimized. The problem just described is called the *optimal execution problem*.

There is a large number of empirical studies on market microstructure and price impact. See for example Bouchaud et al. [2009] and Gatheral et al. [2011] for an overview. The empirical observations led to the following classification of market impacts.

- The *instantaneous* or *temporary impact* (sometimes also called *slippage*) only affects the actual order and then dissipates immediately without memory effect.
- The *transient impact* is caused by temporary imbalances between supply and demand as a result of the actual order. This imbalance causes a temporary deviation from the equilibrium. Due to the resiliency effect of prices, the transient impact is significant for a certain period after the placement of the order but finally vanishes.
- The *permanent impact* denotes the changes in prices that persist during the entire period of the trading activity and is caused by the investor's trades due to, e.g., information on fundamentals revealed by a large order.

Among the first to investigate the problem of optimal liquidation were, for example, Bertsimas and Lo [1998], Huberman and Stanzl [2004] and Almgren and Chriss (see Almgren and Chriss [1999] or Almgren and Chriss [2000]) with discrete-time models. In all these references the dynamics of the price processes are additive random walks affected by the trading strategy. In Bertsimas and Lo [1998] the impact is proportional to the amount of shares traded. In Huberman and Stanzl [2004], Almgren and Chriss [1999] and Almgren and Chriss [2000] the change in the price is caused by a temporary impact as well as a permanent impact which are both assumed to be linear in the number of shares traded. Almgren [2003] considers a continuous time approximation of this approach, in which he uses non-linear and random temporary market impact functions. The distinction between permanent and temporary impact is reasonable as long as we consider discrete-time models and assume that the time between individual orders is sufficiently long. But on a finer time scale to the point of continuous-time trading, it is more realistic to consider a transient impact that decays over a certain time instead of a temporary impact that vanishes instantaneously. This means that by considering transient impact one assumes to have resilience of stock prices after the order placement. Among the first to introduce a transient impact was Obizhaeva and Wang [2005]. Obizhaeva and Wang [2005] assume the existence of a limit order book through which the orders of the large trader are executed. In this setting, the cost of an execution strategy depends on the density of the number of shares being offered at each price and the resilience of the order book. The resilience of the order book is assumed to be exponential. This means that the transient impact on the equity price - which at the moment the order is executed is assumed to be linear in the number of shares traded - decays exponentially in time. As in the models by Almgren et. al, Obizhaeva and Wang [2005] in addition assume a linear permanent impact.

We see that in these just cited "first generation" models, the trading impact is modelled always linearly in the number of shares traded. There are good reasons for this, since Huberman and Stanzl [2004] show that the permanent price impact must be linear to exclude price manipulation strategies. Here, in analogy with the definition of Huberman and Stanzl, a price manipulation is a round-trip trade (i.e., starting with zero shares, buying and selling to end with zero shares) whose expected cost is negative. The principle of *no-dynamic-arbitrage* introduced by Gatheral [2010] states that price manipulation is not possible. Gatheral [2010] considers a continuous-time model with temporary, transient, and permanent impacts and analyses the connection between the decay of the price impact and the possibility of price manipulation strategies. It is shown that a model that combines a nonlinear price impact with a transient impact of exponential decay admits price manipulation. In Gatheral et al. [2011] this statement is generalized and it is proved that any model with a nonlinear market impact function and a decay function that is nonsingular at time zero admits price manipulation.

In Alfonsi et al. [2012] another kind of arbitrage possibility is studied, that of the *transaction-triggered price manipulation strategy.* Citing Alfonsi et al., a market impact model admits transaction-triggered price manipulation if the expected costs of a sell (buy) program can be decreased by intermediate buy (sell) trades. Note that transaction-triggered price manipulation can exist in models that do not admit standard price manipulation in the sense of the Huberman and Stanzl. In Alfonsi et al. [2012], it is proved that (under the assumption of linear price impact) there are no transaction-triggered price manipulation strategies if the resilience functions is convex. Therefore, the authors include in addition to the exponential resilience, a linear resilience, a power-law resilience, and a Gaussian resilience.

In this thesis, we will formulate the optimal execution problem as a stochastic optimal control problem. This means that we will consider a continuous-time model where the

stock price is described by a stochastic process. As is common in the literature for optimal execution models, we will assume that the stock price behaves like a driftless Brownian motion without the investor's trading activity with the consequence that the stock price may become negative. However, since in reality even very large asset positions are typically liquidated within a few days or even hours the probability of negative prices is negligible.
The trader has some control over the stock price by choosing a trading speed at any time during the fixed given trading period. The chosen trading speed enters the drift of the stock price process in the form of a permanent and a transient impact. In order to profit from the nice properties of the exponential function, we will assume an exponential resilience of the transient impact in order to hold the model mathematically tractable. We will assume a linear price impact, since by Gatheral [2010] an exponential decay of market impact is not compatible with a nonlinear one. The aim of the stochastic control problem we consider will be to model the selling of a certain amount of shares within a given time horizon in such a way that the expected average selling price is maximized. Clearly, the average selling price is equal to the integrated product of trading speed and stock price over the trading horizon. Therefore, we consider a stochastic control problem with the following properties:

- the stock price is modeled by a Brownian motion with a drift that is linear in the control variable, i.e., the trading speed;
- to model the impact of past trading decisions the stock price process contains some exponential delay effect;
- the cost functional of the control problem will be linear in the control variable.

In Chapter 4, we will see that the above properties lead to a stochastic control problem with two difficulties. Due to the exponential delay in the control variable, we will obtain a stochastic control problem with a degenerate diffusion and due to the linearity with respect to the control variable we will obtain under appropriate additional assumptions an optimal solution with "bang-bang" character. This means that the optimal trading strategy is not continuous but switches abruptly between extreme states. This discontinuity makes the solution of the problem very difficult.
If the intention of this thesis was only to make a contribution to the question of optimal execution, one could avoid the named problems by changing the model by not requiring linearity in the control variable. For that, one could for example choose another cost functional. Almgren [2003], Almgren and Chriss [1999] and Almgren and Chriss [2000] for example incorporate the risk into the execution problem by minimizing an objective function which is a linear combination of expected cost and risk. Their paper therefore deals with the more general maximization of the expected revenue of trading with a suitable penalty for the uncertainty of revenue. In Schied and Sch"oneborn [2007] the trader chooses a trading strategy such that the expected *utility* of the portfolio value is maximized. Within this work, different types of utility functions and their corresponding optimal solutions are discussed. Both approaches could help to obtain a cost functional

that is nonlinear with respect to the control variable.
Instead of varying the cost functional, one could consider alternative formulations of the stock price process and the trading impact on the price process. One possibility, for example, could be to choose a nonlinear impact together with a resilience that is non-exponential. The stock price process would lose its linearity with respect to the trading speed. But such a model would be very difficult to handle. To our knowledge, besides the work of Alfonsi et al. [2010], whose dynamics is derived from an underlying limit order book, there are no references with solutions to this kind of problems.
Another alternative could be not to consider a purely continuous trading strategy but to allow in addition for discrete trading activities. This is done in Obizhaeva and Wang [2005] or in Predoiu et al. [2011]. Or one could consider a singular control process instead of a continuous one. This is done for example in Fruth [2011], who expresses the optimal execution problem as a singular control problem.
There are obviously many alternative formulations of the optimal execution problem, and a lot of different approaches are treated in recent work. As described above, our formulation of the problem leads to a mathematically challenging problem. Besides the optimal solution of the execution problem, we therefore also provide the existence of a solution to a stochastic bang-bang problem with degenerate diffusion. This objective will even be the main topic of this thesis.

The thesis is organized as follows. Chapter 2 contains some background material needed throughout. In Chapter 3, we will give a short introduction to the theory of stochastic optimal control. In Chapter 4, we will discuss in detail - encouraged by the optimal execution problem - two special cases of control problems. On the one hand, we consider a stochastic control problem, where the state variable has an exponential delay in the control variable. On the other hand, we treat a stochastic control problem with coefficients that are linear in the control variable. We will show that in the first case the stochastic control problem can be transferred into another control problem that has no delay but contains a degenerate diffusion process. In the second case, we will show that the optimal control process is a bang-bang process. Since as explained in this section, it is reasonable to model the optimal execution problem by a stochastic control problem that combines both special cases in Chapter 5, we will consider a general formulated stochastic control problem with coefficients that are linear in the control variable and a possibly degenerate diffusion. We will show that an optimal (bang-bang) solution exists and that it is possible to approximate it by non-singular control problems. Finally, in Chapter 6, we will apply the results to the optimal execution problem, and illustrate them by numerical simulations.

2. Preliminaries

2.1. Convergence of Probabilities and Stochastic Processes

In this section, we will recall the concept and characterization of weak convergence of probability measures. The concept is essential for proving some existence results in stochastic analysis and stochastic control. The results here will be presented without proofs. The details can be found for example in Ethier and Kurtz [1986] and Ikeda and Watanabe [1989].
Let (U, ρ) be a separable metric space and $\mathcal{B}(U)$ the corresponding Borel σ-field. Denote by $\mathcal{P}(U)$ the set of all probability measures defined on $(U, \mathcal{B}(U))$.

Definition 2.1.1. A sequence $(P_n)_{n\in\mathbb{N}} \subseteq \mathcal{P}(U)$ is said to be *weakly convergent* to $P \in \mathcal{P}(U)$ if for any $f \in C_b(U)$ we have

$$\lim_{n\to\infty} \int_U f(u)dP_n(u) = \int_U f(u)dP(u),$$

where $C_b(U)$ is the space of bounded and continuous functions on U.

Definition 2.1.2. A set $\Lambda \subseteq \mathcal{P}(U)$ is said to be

1. *relatively compact* if any sequence $(P_n)_{n\in\mathbb{N}} \subseteq \Lambda$ contains a weakly convergent subsequence;

2. *compact* if Λ is relatively compact and closed;

3. *tight* if for any $\epsilon > 0$ there is a compact set $K \subseteq U$ such that $\inf_{P\in\Lambda} P(K) \geq 1-\epsilon$.

Theorem 2.1.3. *Let $\Lambda \subseteq \mathcal{P}(U)$. Then the following statements hold.*

1. *If Λ is tight, then Λ is relatively compact in $\mathcal{P}(U)$.*

2. *If (U, ρ) is complete, the converse of 1. holds. Namely, if Λ is relatively compact in $\mathcal{P}(U)$, then Λ is tight.*

Proof. For a proof we refer to Ikeda and Watanabe [1989], Theorem 1.2.6.

Corollary 2.1.4. *If (U, ρ) is compact, then any $\Lambda \subseteq \mathcal{P}(U)$ is tight and relatively compact. In particular, $\mathcal{P}(U)$ is compact.*

Proof. For a proof see Parthasarathy, Theorem 2.3.2 and Theorem 2.6.4.

□

Consider now a probability space $(\Omega, \mathcal{F}, P)$ with a random variable $X : (\Omega, \mathcal{F}, P) \to (U, \rho)$. Then $P_X \in \mathcal{P}(U)$ denotes the probability law induced by X.

Definition 2.1.5. We say that a sequence of random variables $(X^n)_{n\in\mathbb{N}}$ with $X^n : (\Omega, \mathcal{F}, P) \to (U, \rho)$ is tight if $(P_{X^n})_{n\in\mathbb{N}}$ is tight.

The following convergence concepts are important.

Definition 2.1.6. Let $X^n, X : (\Omega, \mathcal{F}, P) \to (U, \rho)$, $n \in \mathbb{N}$, be random variables. We say that $(X^n)_{n\in\mathbb{N}}$ converges to X almost surely (a.s.) if

$$P\left(\lim_{n\to\infty} \rho(X^n, X) = 0\right) = 1,$$

that $(X^n)_{n\in\mathbb{N}}$ converges to X in probability if for any $\epsilon > 0$,

$$\lim_{n\to\infty} P\left(\rho(X^n, X) > \epsilon\right) = 0,$$

and that $(X^n)_{n\in\mathbb{N}}$ converges to X in law (or in distribution) and write $X^n \xrightarrow{\mathcal{D}} X$ if

$$P_{X^n} \to P_X \text{ weakly as } n \to \infty.$$

Remark 2.1.7. The different definitions of convergence are connected in the way that for a sequence of random variables almost sure convergence implies convergence in probability and convergence in probability implies convergence in law.

Consider next a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$. In the following we say the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ satisfies the *usual conditions*, if $(\Omega, \mathcal{F}, P)$ is complete, $\mathcal{F}_0$ contains all the P-null sets in $\mathcal{F}$, and $(\mathcal{F}_t)_{t\geq 0}$ is right continuous. In what follows we denote by $X = (X_t)_{t\geq 0}$ a d-dimensional stochastic process.

Definition 2.1.8. The process X is called *progressively measurable* with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if for all $T \geq 0$ the function $(\omega, t) \to X_t(\omega)$ considered as a map between $\Omega \times [0, T]$ and $\mathbb{R}^d$ is measurable with respect to $\mathcal{F}_T \times \mathcal{B}([0, T])$ and $\mathcal{B}(\mathbb{R}^d)$.

Let now X and X' be two d-dimensional stochastic processes. We say that the two processes have the same law if and only if all their finite dimensional distributions coincide, i.e.,

$$P\left(X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n\right) = P\left(X'_{t_1} \in B_1, \ldots, X'_{t_n} \in B_n\right),$$

for all $n \in \mathbb{N}$, $t_1, \ldots, t_n > 0$ and Borel sets $B_1, \ldots, B_n$. We write in this case $X \overset{\mathcal{L}}{\approx} X'$. Two processes which have the same law are identically distributed and we write $P_X = P_{X'}$. Let us now introduce the notion of tightness for random variables. The following criterion for tightness of random variables goes back to Aldous.

Theorem 2.1.9 (Aldous Criterion)**.** *Let $(X^n)_{n\in\mathbb{N}}$ be a sequence of d-dimensional continuous processes. The sequence $(P_{X^n})_{n\in\mathbb{N}}$ is tight if and only if for every $T > 0$ and $\epsilon > 0$*

$$\lim_{c\to\infty} \sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} P\left(|X_t^n| > c\right) = 0, \tag{2.1}$$

$$\lim_{\theta \downarrow 0} \sup_{n\in\mathbb{N}} \sup_{t,s} \{P(|X_t^n - X_s^n| > \epsilon) : 0 \le t, s \le T, |t-s| \le \theta\} = 0. \tag{2.2}$$

Proof. For a proof we refer to Karatzas and Shreve [1991], Theorem 4.10, p. 63.

□

The following theorem gives some tool to check in a comfortable way whether a sequence of processes fulfills the Aldous criterion.

Theorem 2.1.10. *Let $(X^n)_{n\in\mathbb{N}}$ be a sequence of d-dimensional continuous processes over $[0,T]$ satisfying the following two conditions.*

- *There exists a positive constant γ such that $\sup_{n\ge 1} E[|X_0^n|^\gamma] < \infty$,*
- *there exist positive constants α, β and K such that $\sup_{n\ge 1} E[|X_t^n - X_s^n|^\alpha] \le K|t-s|^{1+\beta}$ for all $t,s\in[0,T]$.*

Then the statements of Theorem 2.1.9 hold for $(X^n)_{n\in\mathbb{N}}$.

In general, it is often necessary to work with *continuous* stochastic processes. Therefore, the following theorem is very helpful, because it gives a sufficient condition for the existence of a continuous modification of a d-dimensional stochastic process.

Corollary 2.1.11 (Kolmogorov's Continuity Theorem)**.** *Let X be a d-dimensional stochastic process over $[0,T]$ such that for some positive constants K, α and β*

$$E[|X_t - X_s|^\alpha] \le K|t-s|^{1+\beta}$$

for all $t,s\in[0,T]$. Then there exists a d-dimensional continuous process $\hat{X}$ such that for every $t\in[0,T]$, $P[X_t = \hat{X}_t] = 1$.

See Ikeda and Watanabe [1989] (pp. 17-20) for proofs of Theorem 2.1.10 as well as Corollary 2.1.11.

The next theorem shows that a weakly convergent sequence of probability measures can be represented as the distribution of a pointwise convergent sequence of random variables defined on a common probability space.

Theorem 2.1.12 (Skorohod's Representation Theorem)**.** *Let (U,δ) be a separable metric space and assume the probability measures $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{P}(U)$ converge weakly to $\mu\in\mathcal{P}(U)$. Then there exists a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ on which are defined U-valued random variables $(\hat{X}^n)_{n\in\mathbb{N}}$ and $\hat{X}$ such that for all Borel sets $B\in\mathcal{B}(U)$ and all $n\in\mathbb{N}$*

$$\hat{P}(\hat{X}^n \in B) = \mu_n(B), \quad \hat{P}(\hat{X}\in B) = \mu(B)$$

and such that

$$\lim_{n\to\infty} \hat{X}^n = \hat{X} \quad a.s.$$

Proof. For a proof see Ethier and Kurtz [1986], Theorem 1.8, p. 102.

□

The following Corollary applies Skorohod's Theorem to a tight sequence of stochastic processes.

Corollary 2.1.13. *Let $(X^n)_{n\in\mathbb{N}}$ be a sequence of d-dimensional continuous processes and assume that $(P_{X^n})_{n\in\mathbb{N}}$ is tight. Then there exists a subsequence $(n_k)_{k\in\mathbb{N}}$, a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ and d-dimensional continuous processes $\hat{X}^{n_k}$, $k \in \mathbb{N}$, and $\hat{X}$ defined on it such that*

- $X^{n_k} \overset{\mathcal{L}}{\approx} \hat{X}^{n_k}$ *for all* $k \in \mathbb{N}$,
- $\hat{X}^{n_k}$ *converges to* $\hat{X}$ *almost surely as* $k \to \infty$.

Proof. Since $(P_{X^n})_{n\in\mathbb{N}}$ is assumed to be tight, it follows from 2.1.3 that $(P_{X^n})_{n\in\mathbb{N}}$ is relatively compact. Consequently, there is a subsequence of $(P_{X^n})_{n\in\mathbb{N}}$ that is weakly convergent. The rest of the proof follows from Theorem 2.1.12.

□

For later use we will give the following definition of weak convergence of stochastic processes.

Definition 2.1.14. Let $(X^n)_{n\geq 1}$ be a sequence of stochastic processes defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. We assume $1 \leq p < \infty$ and let p' be the conjugate exponent, i.e.,

$$\frac{1}{p} + \frac{1}{p'} = 1 \quad (p' \equiv \infty \text{ if } p = 1).$$

A sequence $(X^n)_{n\geq 1} \subset L^p([0,T] \times \Omega; \mathbb{R}^d)$ converges weakly to $X \in L^p([0,T] \times \Omega; \mathbb{R}^d)$ if and only if for all $Y \in L^{p'}([0,T] \times \Omega; \mathbb{R}^d)$

$$\lim_{n\to\infty} \int_{[0,T]\times\Omega} Y_s(\omega) X^n_s(\omega) ds dP(\omega) \to \int_{[0,T]\times\Omega} Y_s(\omega) X_s(\omega) ds dP(\omega).$$

2.2. Brownian Motion

The following definition of a process called *Brownian Motion* is standard.

Definition 2.2.1. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space satisfying the usual conditions. The process W is called a d-dimensional standard $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion over $[0, \infty)$ if

- $W_0 = 0$ a.s.,

- W is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$, and $W_t - W_s$ is independent of $\mathcal{F}_s$ for every $0 \leq s \leq t$,

- $W_t - W_s$ is $N(0,(t-s)I_d)$ (i.e., normal with mean vector 0 and covariance matrix $(t-s)I_d$, where I_d denotes the d-dimensional identity matrix) for every $0 \leq s \leq t$,

- W has continuous sample paths in $\mathbb{R}^d$.

We will sometimes assume for convenience that the filtration of the considered filtered probability space is the P-augmented natural filtration generated by W, that is

$$\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{N}, \quad t \geq 0,$$

where

$$\mathcal{F}_t^W \equiv \sigma\{W_s, 0 \leq s \leq t\}, \quad t \geq 0,$$

and

$$\mathcal{N} = \{A \subset \Omega | \text{ there exists a } B \in \mathcal{F} \text{ such that } P(B) = 0 \text{ and } A \subset B\}.$$

Here, the symbol $\mathcal{F}_t^W \vee \mathcal{N}$ denotes the smallest σ-algebra containing $\mathcal{F}_t^W$ and $\mathcal{N}$.
In this case the five-tuple $\nu \equiv (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P, W)$ will be referred to as a d-dimensional *Brownian stochastic basis.*
We recall the definition of a martingale.

Definition 2.2.2. Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the usual conditions, an $(\mathcal{F}_t)_{t\geq 0}$-adapted real valued process $M = (M_t)_{t\geq 0}$ is said to be an $(\mathcal{F}_t)_{t\geq 0}$-martingale if the following holds.

- $E[|M_t|] < +\infty$ for all $t \geq 0$,

- $E[M_t|\mathcal{F}_s] = M_s$, a.s. for all $0 \leq s \leq t$.

An $(\mathcal{F}_t)_{t\geq 0}$-adapted $\mathbb{R}^d$-valued process $M = (M_t)_{t\geq 0}$ is said to be an $(\mathcal{F}_t)_{t\geq 0}$-martingale if each of its d components is an $(\mathcal{F}_t)_{t\geq 0}$-martingale.

Definition 2.2.3. Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the usual conditions and an $(\mathcal{F}_t)_{t\geq 0}$-martingale $M = (M_t)_{t\geq 0}$, we say that M is *square-integrable* if $E[M_t^2] < \infty$ for every $t \geq 0$.

Definition 2.2.4 (Local Martingale)**.** A real-valued process $M = (M_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ is an $(\mathcal{F}_t)_{t\geq 0}$-local martingale if there exists a non-decreasing sequence of $(\mathcal{F}_t)$-stopping times $(\tau_n)_{n\in\mathbb{N}}$ with $\tau_n \to \infty$ a.s. such that $M_\cdot^{\tau_n} \equiv M_{\cdot\wedge\tau_n}$ is an $(\mathcal{F}_t)_{t\geq 0}$-martingale. If, in addition, $M_\cdot^{\tau_n}$ is a square integrable martingale for each $n \in \mathbb{N}$, then we call M a *locally square integrable* $(\mathcal{F}_t)_{t\geq 0}$*-martingale.*

Remark 2.2.5. Every martingale is a local martingale, but the converse is not true.

Let $T > 0$ and $\nu = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P, W)$ be a Brownian stochastic basis. Then we introduce the following sets.

$$\begin{cases} \mathcal{M}^2[0,T] = \Big\{M | M \in L^2_\nu([0,T];\mathbb{R}) \text{ and M is a right-continuous } (\mathcal{F}_t)_{t\geq 0}\text{-martingale} \\ \qquad\qquad\qquad \text{with } M_0 = 0, \text{ P-a.s.}\big\}, \\ \mathcal{M}^2_c[0,T] = \{M | M \in \mathcal{M}^2[0,T] \text{ and } t \mapsto M_t \text{ is continuous, P-a.s.}\}, \\ \mathcal{M}^{2,loc}[0,T] = \{M | M \in L^2_\nu([0,T];\mathbb{R}) \text{ and there exists a sequence of non-decreasing} \\ \qquad\qquad\qquad \text{stopping times } (\tau_n)_{n\in\mathbb{N}} \text{ with } P(\lim_{n\to\infty} \tau_n \geq T) = 1 \text{ and} \\ \qquad\qquad\qquad M^{\tau_n}_\cdot = M_{\cdot\wedge\tau_n} \in \mathcal{M}^2[0,T], n \in \mathbb{N}\}, \\ \mathcal{M}^{2,loc}_c[0,T] = \Big\{M | M \in \mathcal{M}^{2,loc}[0,T] \text{ and } t \mapsto M_t \text{ is continuous, P-a.s.}\Big\}. \end{cases}$$

Definition 2.2.6 (Quadratic Variation)**.** Let $M \in \mathcal{M}^{2,loc}[0,T]^d$. A process $(A_t)_{t\in[0,T]}$ is called the *quadratic variation (process)* of M if $A_0 = 0$ a.s. and $M_t M_t^* - A_t$, $t \in [0,T]$, is an $(\mathcal{F}_t)_{t\geq 0}$-martingale. We denote it by $[M]_t \equiv A_t$, $t \in [0,T]$.

Definition 2.2.7 (Cross-Variation)**.** For any two $X, Y \in \mathcal{M}^{2,loc}[0,T]^d$, we define their cross-variation process $[X,Y]$ by

$$[X,Y]_t \equiv \frac{1}{4}\left([X+Y]_t - [X-Y]_t\right), \quad t \in [0,T],$$

and observe that $XY^* - [X,Y]$ is a martingale.

Remark 2.2.8. With help of the uniqueness argument in the Doob-Meyer Composition (see Karatzas and Shreve [1991], Chapter 1, Theorem 4.10), it can be shown that $[X,Y]$ is, up to indistinguishability, the only process A such that $XY^* - A$ is a martingale.

Remark 2.2.9. Note that $[X,X] = [X]$.

Theorem 2.2.10 (Levy's Theorem)**.** *Let $X_t = (X^1_t, X^2_t, \cdots, X^d_t)$, $t \geq 0$, be a continuous, $(\mathcal{F}_t)_{t\geq 0}$-adapted process in $\mathbb{R}^d$ such that, for every component $1 \leq k \leq d$ the process*

$$M^k_t \equiv X^k_t - X^k_0, \quad t \geq 0,$$

is a continuous local martingale relative to $(\mathcal{F}_t)_{t\geq 0}$, and for every $1 \leq k, j \leq d$ the cross-variations are given by

$$\left[M^k, M^j\right]_t = \delta_{kj} t, \quad t \geq 0, \tag{2.3}$$

where δ_{kj} is the Kronecker delta.
Then X is a d-dimensional Brownian motion.

Proof. See Karatzas and Shreve [1991], Chapter 3, Theorem 3.16.

□

Note that property (2.3) characterizes a Brownian motion among *continuous* local martingales. The compensated Poisson process with intensity $\lambda = 1$ provides an example

of a *discontinuous*, square-integrable martingale with $[M]_t = t$, $t \geq 0$. So the assumption of continuity in the latter theorem is essential. It follows that the one-dimensional Brownian motion is the unique member of $\mathcal{M}_c^{loc}$ whose quadratic variation at time t is t.
The following result will be useful.

Lemma 2.2.11. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space satisfying the usual conditions. A continuous d-dimensional process W is a d-dimensional $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion if and only if for any $0 \leq s \leq t < \infty$,*

$$\begin{cases} E\left[W_t - W_s | \mathcal{F}_s\right] = 0, \quad P - a.s., \\ E\left[(W_t - W_s)(W_t - W_s)^* | \mathcal{F}_s\right] = (t-s)I_d, \quad P - a.s. \end{cases} \tag{2.4}$$

Proof. "$\Rightarrow$": Let W be a d-dimensional Brownian motion, then $P_{W_t - W_s} = N(0, (t - s)I_d)$. Consequently, from the independent increment property we get

$$0 = E\left[W_t - W_s\right] = E\left[W_t - W_s | \mathcal{F}_s\right]$$

and

$$\begin{aligned} (t-s)I_d &= Var[W_t - W_s] \\ &= E[(W_t - W_s)(W_t - W_s)^*] - E[W_t - W_s]E[W_t - W_s]^* \\ &= E[(W_t - W_s)(W_t - W_s)^* | \mathcal{F}_s] - E[W_t - W_s | \mathcal{F}_s]E[W_t - W_s | \mathcal{F}_s]^* \\ &= E[(W_t - W_s)(W_t - W_s)^* | \mathcal{F}_s]. \end{aligned}$$

"$\Leftarrow$": For every $0 \leq i, j \leq d$ and $0 \leq s \leq t < \infty$ the second equation of (2.4) implies that

$$E\left[(W_t^i - W_s^i)(W_t^j - W_s^j) | \mathcal{F}_s\right] = \delta_{ij}(t-s). \tag{2.5}$$

Define for every component $1 \leq k \leq d$ of W the process

$$M_t^k \equiv W_t^k - W_0^k, \quad t \geq 0.$$

It follows for $0 \leq i, j \leq d$ and $0 \leq s \leq t < \infty$

$$M_t^i - M_s^i = (W_t^i - W_0^i) - (W_s^i - W_0^i) = W_t^i - W_s^i.$$

Therefore, equation (2.5) is equivalent to

$$E\left[(M_t^i - M_s^i)(M_t^j - M_s^j) | \mathcal{F}_s\right] = \delta_{ij}(t-s). \tag{2.6}$$

From the first equation in (2.4) if follows that W is an $(\mathcal{F}_t)_{t\geq 0}$-martingale. Consequently, M^k is an $(\mathcal{F}_t)_{t\geq 0}$-martingale as well for every $1 \leq k \leq d$.
Consider next for $0 \leq i, j \leq d$ and $0 \leq s \leq t < \infty$

$$E\left[(M_t^i - M_s^i)(M_t^j - M_s^j) | \mathcal{F}_s\right]$$

$$
\begin{aligned}
&= E\left[M_t^i M_t^j - E\left[M_t^i|\mathcal{F}_s\right] M_s^j - M_s^i E\left[M_t^j|\mathcal{F}_s\right] + M_s^i M_s^j|\mathcal{F}_s\right] \\
&= E\left[M_t^i M_t^j - M_s^i M_s^j - M_s^i M_s^j + M_s^i M_s^j|\mathcal{F}_s\right] \\
&= E\left[M_t^i M_t^j - M_s^i M_s^j|\mathcal{F}_s\right].
\end{aligned}
$$

Together with equation (2.6) we obtain for $0 \le i,j \le d$ and $0 \le s \le t < \infty$

$$E\left[M_t^i M_t^j - M_s^i M_s^j|\mathcal{F}_s\right] = \delta_{ij}(t-s).$$

Since the last equation is equivalent to

$$E\left[M_t^i M_t^j - \delta_{ij}t|\mathcal{F}_s\right] = M_s^i M_s^j - \delta_{ij}s,$$

it follows that $M_t^i M_t^j - \delta_{ij}t$, $t \ge 0$, is an $(\mathcal{F}_t)_{t\ge 0}$-martingale for every $0 \le i,j \le d$. From the uniqueness of the variation process (see Remark 2.2.8) it follows

$$\left[M^i, M^j\right]_t = \delta_{ij}t, \quad t > 0,$$

and it follows from Theorem 2.2.10 that W is a d-dimensional Brownian motion.

□

We will need the preceding result in Chapter 5. There we will have to prove that a certain continuous process is a Brownian motion corresponding to a given filtration. To do this, we will make use of the following proposition. The proof can be taken from Yong and Zhou [1999], Proposition 1.12 in Chapter 1.

Proposition 2.2.12. *For $i \in \mathbb{N}$ let $d_i \in \mathbb{N}$ and let $\xi_i : (\Omega, \mathcal{F}) \to (\mathbb{R}^{d_i}, \mathcal{B}(\mathbb{R}^{d_i}))$ be a sequence of random variables. Let $\mathcal{G} \equiv \sigma(\xi_i : i \in \mathbb{N})$ and $X \in L^2_\nu(\Omega; \mathbb{R}^d)$. Then $E(X|\mathcal{G}) = 0$ if and only if $E(g(\xi_1, \xi_2, ..., \xi_i)X) = 0$ for any i and any $g \in C_b(\mathbb{R}^N)$, where $N = \sum_{j=1}^i d_j$.*

Remark 2.2.13. If we can show for a continuous d-dimensional process $(\xi_t)_{t\ge 0}$ that for any $0 \le t \le s < \infty$, any $l \in \mathbb{N}$ and any $g \in C_b(\mathbb{R}^{ld})$

$$E\left[g(y)(\xi_s - \xi_t)\right] = 0, \tag{2.7}$$

and

$$E\left[g(y)(\xi_s - \xi_t)(\xi_s - \xi_t)^*\right] = (s-t)I_d, \tag{2.8}$$

where

$$y = \{\xi(t_i), 0 \le t_1 \le t_2 \le ... \le t_l \le s\}$$

for any partition $0 \le t_1 \le t_2 \le ... \le t_l \le s$ of $[0, s]$. Then Proposition 2.2.13 implies that ξ is a Brownian motion since (2.7) leads to the first equation and (2.8) to the second equation in (2.4).

It can be shown that a Brownian motion can be constructed by using so called *Borel cylinder sets* (see for example Karatzas and Shreve [1991], pp. 49-56). We will later see that the use of cylinders sets is useful for our interests as well. Therefore, we give here the following definition.

Definition 2.2.14 (Borel cylinder)**.** Let $C([0,\infty);\mathbb{R}^d)$ denote the set of continuous functions on $[0,\infty)$ taking values in $\mathbb{R}^d$. Then a set $B \subset C([0,\infty);\mathbb{R}^d)$ is called a *Borel cylinder* if there exists $j \in \mathbb{N}$ and $0 \leq t_1 < \cdots < t_j < \infty$ and $E \in \mathcal{B}(\mathbb{R}^{jd})$ such that

$$B = \left\{ \zeta \in C([0,\infty);\mathbb{R}^d) | (\zeta(t_1), \zeta(t_2), ..., \zeta(t_j)) \in E \right\}. \tag{2.9}$$

We let $\mathbf{C}$ be the set of all Borel cylinders in $C([0,\infty);\mathbb{R}^d)$ of the form (2.9).

We will need the following lemma.

Lemma 2.2.15. *The σ-field $\sigma(\mathbf{C})$ generated by $\mathbf{C}$ coincides with the Borel σ-field $\mathcal{B}(C([0,\infty);\mathbb{R}^d))$.*

Proof. See Ikeda and Watanabe [1989], Chapter 1, Proposition 4.1.

□

2.3. Martingale Representation Theorems

Let in the following T > 0 be a finite time horizon. Assume that a 1-dimensional stochastic basis $\nu = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P, W)$ is given. For some function $f \in L^2_\nu([0,T];\mathbb{R})$ consider the one-dimensional Itô integral

$$M_t = \int_0^t f(s) dW_s, \quad 0 \leq t \leq T.$$

For details about the construction and definition of Itô integrals, we refer to Ikeda and Watanabe [1989], pp. 45-51, or Karatzas and Shreve [1991], pp. 129-141.
Whereas it is easy to see that an Itô integral of the above form defines a martingale, the question whether every martingale can be represented as an Itô integral is more difficult and is answered by so called *martingale representation theorem.*
For a general formulation, we will use the following notation for higher-dimensional Itô integrals.
Let W be an d-dimensional Brownian motion and $f = (f_1, ..., f_d) \in L^2_\nu([0,T];\mathbb{R}^d)$. Then for $j = 1, 2, ..., d$ and $0 \leq t \leq T$, $\int_0^t f_j(s) dW_s^j$ is well-defined and we set

$$\int_0^t \langle f(s), dW_s \rangle \equiv \sum_{j=1}^d \int_0^t f_j(s) dW_s^j, \quad 0 \leq t \leq T. \tag{2.10}$$

For some matrix valued function $\sigma = (\sigma_{ij}) \in L^2_\nu([0,T];\mathbb{R}^{d\times\ell})$ with $d,\ell \in \mathbb{N}$, we define

$$\int_0^t \sigma(s)dW_s \equiv \begin{pmatrix} \sum_{j=1}^{\ell} \int_0^t \sigma_{1j}(s)dW_s^j \\ \vdots \\ \sum_{j=1}^{\ell} \int_0^t \sigma_{dj}(s)dW_s^j \end{pmatrix}, \quad 0 \le t \le T. \tag{2.11}$$

Note that (2.10), resp. (2.11) represent elements in $\mathcal{M}_c^2[0,T]$, resp. $(\mathcal{M}_c^2[0,T])^d$. The following theorem is concerned with the representation of a martingale with respect to a fixed Brownian motion.

Theorem 2.3.1. *Let $\nu = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P, W)$ be a d-dimensional stochastic basis. Let $X \in \mathcal{M}^{2,loc}[0,T]$. Then there exists a unique $\varphi \in L^2_\nu([0,T];\mathbb{R}^d)$ such that*

$$X_t = \int_0^t \langle \varphi(s), dW_s \rangle, \quad t \in [0,T], P-a.s.$$

Proof. See Ikeda and Watanabe [1989], Chapter II, Theorem 6.6.

□

The latter theorem also holds under weaker assumptions. However, it is generally necessary to extend the given probability space in order to guarantee the existence of a Brownian motion. Before stating this result, we shall first make precise the notion of an extension of a probability space.

Definition 2.3.2. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$ and $(\hat{\Omega}, \hat{\mathcal{F}}, \left(\hat{\mathcal{F}}_t\right)_{t\ge 0}, \hat{P})$ be two filtered probability spaces satisfying the usual conditions. The latter is called an *extension* of the former if there exists a random variable $\pi : (\hat{\Omega}, \hat{\mathcal{F}}) \to (\Omega, \mathcal{F})$ such that
(i) $\pi^{-1}(\mathcal{F}_t) \subseteq \hat{\mathcal{F}}_t$ for all $t \ge 0$,
(ii) $P = \hat{P} \circ \pi^{-1}$ and
(iii) for any $X \in L^\infty_\nu(\Omega)$,

$$\hat{E}[\hat{X}(\hat{\omega})|\hat{\mathcal{F}}_t](\hat{\omega}) = E[X|\mathcal{F}_t](\pi\hat{\omega}), \quad P\text{-a.a. } \hat{\omega} \in \hat{\Omega},$$

where we set $\hat{X}(\hat{\omega}) \equiv X(\pi\hat{\omega})$ for $\hat{\omega} \in \hat{\Omega}$.

Now we are able to state the second martingale representation theorem.

Theorem 2.3.3. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$ be a filtered probability space satisfying the usual conditions. For $d,\ell \in \mathbb{N}$, let $M \in \mathcal{M}_c^{2,loc}[0,T]^d$ and $\sigma \in L^{2,loc}_\nu([0,T];\mathbb{R}^{d\times\ell})$ with $\sigma\sigma^* \in L^{1,loc}_\nu([0,T];\mathbb{R}^{d\times d})$. If*

$$\langle M \rangle_t = \int_0^t \sigma(s)\sigma(s)^* ds,$$

then there exists an extension $(\hat{\Omega}, \hat{\mathcal{F}}, \left(\hat{\mathcal{F}}_t\right)_{t\ge 0}, \hat{P})$ of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$ on which there *lives an ℓ-dimensional $\left(\hat{\mathcal{F}}_t\right)_{t\geq 0}$-Brownian motion W such that*

$$M_t = \int_0^t \sigma(s) dW_s, \quad t \geq 0.$$

Proof. See Ikeda and Watanabe [1989], Chapter II, Theorem 7.1'.

□

2.4. Stochastic Differential Equations

In this section we will briefly review d-dimensional *Stochastic Differential Equations* (SDEs).
In what follows, we assume $x \in \mathbb{R}^d$ to be deterministic, let $t \geq 0$ and assume $T < \infty$ to be fixed. Let W be an ℓ-dimensional Brownian motion and consider the functions $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times\ell}$. Then, we consider a family of SDEs of the form

$$\begin{cases} dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dW_s, \quad s \in [t,T], \\ X_t^{t,x} = x. \end{cases} \tag{2.12}$$

Equation (2.12) can be written equivalently as a stochastic integral equation of the form

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dW_r, \quad s \in [t,T]. \tag{2.13}$$

In the following, we will distinguish between two different types of solution concepts for SDEs. These are concepts of *strong solutions* and *weak solutions*.

2.4.1. Strong Solutions

We start with a definition of strong solutions that follows Karatzas and Shreve [1991].

Definition 2.4.1. A *strong solution* of the stochastic differential equation (2.12) on the given ℓ-dimensional Brownian stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W)$, is a family of processes $(X^{t,x})_{(t,x)\in[0,T]\times\mathbb{R}^d}$ with continuous sample paths such that for $(t,x) \in [0,T] \times \mathbb{R}^d$ the following properties hold

(i) $X^{t,x} = \left(X_s^{t,x}\right)_{s\in[t,T]}$ is $(\mathcal{F}_s)_{s\geq 0}$-adapted,

(ii) $P(X_t^{t,x} = x) = 1$,

(iii) $\int_t^s \left\{|b(r, X_r^{t,x})| + \left\|\sigma(r, X_r^{t,x})\right\|^2\right\} dr < \infty$, for all $s \in [t,T]$, P-a.s.,

(iv) $X^{t,x}$ satisfies the stochastic integral equation (2.13).

If there is no danger of confusion we will sometimes omit the superscript (t,x) in what follows.

Definition 2.4.2. [Strong Uniqueness] If for any two strong solutions $(X^{t,x})_{(t,x)\in[0,T]\times\mathbb{R}^d}$ and $(\tilde{X}^{t,x})_{(t,x)\in[0,T]\times\mathbb{R}^d}$ of (2.12) defined on a Brownian stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W)$ we have for $t \in [0,T]$ and $x \in \mathbb{R}^d$

$$P(X_s^{t,x} = \tilde{X}_s^{t,x}, s \in [t,T]) = 1,$$

then we say that the strong solution is *unique* or that *strong uniqueness* holds.

We will need the following global Lipschitz and linear growth condition on the coefficients in order to ensure the existence of a strong solution.

Assumption 2.4.3. *Assume there exists a positive constant K such that for all $s \in [0,T]$ and $x, y \in \mathbb{R}^d$ we have*

$$\begin{aligned} |b(s,x) - b(s,y)| + \|\sigma(s,x) - \sigma(s,y)\| &\leq K|x-y|, \\ |b(s,x)| + \|\sigma(s,x)\| &\leq K(1+|x|). \end{aligned}$$

Theorem 2.4.4. *Let $\nu = (\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W)$ be an ℓ-dimensional Brownian stochastic basis. Let b and σ satisfy Assumption 2.4.3. Then there exists a unique strong solution $(X^{t,x})_{(t,x)\in[0,T]\times\mathbb{R}^d}$ to (2.12) and for every $t \in [0,T]$ and $x \in \mathbb{R}^d$ we have $X^{t,x} \in \mathcal{S}^2_\nu([t,T];\mathbb{R}^d)$. Furthermore, for any $p \geq 1$ there exists a constant $C > 0$ such that for any $(t,x) \in [0,T]\times\mathbb{R}^d$*

$$E\left[\sup_{s\in[t,T]} |X_s^{t,x}|^p\right] \leq C(1+|x|^p), \tag{2.14}$$

and for all $(t,x), (t',x') \in [0,T]\times\mathbb{R}^d$, $s' \in [t',T]$ and $s \in [t,T]$,

$$E\left[|X_s^{t,x} - X_{s'}^{t',x'}|^p\right] \leq C\left(|x-x'|^p + (1+|x|+|x'|)^p\left(|t-t'|^{p/2} + |s-s'|^{p/2}\right)\right). \tag{2.15}$$

Proof. For the existence and uniqueness of a solution and for equation (2.14) we refer to Karatzas and Shreve [1991], Theorem 5.2.9. For the proof of equation (2.15) see Kunita [1990], Lemma 4.5.6.

We now establish the Markov property of the solution of a stochastic differential equation of the above type. We say that a process X is *Markovian* if its future depends on the past only through the present. Mathematically, this can be stated as follows.

Definition 2.4.5 (Markov Property)**.** Let X be a progressively measurable process on $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W)$. Then X is said to be Markovian with respect to $(\mathcal{F}_s)_{s\geq 0}$ if, for every bounded measurable function $f: \mathbb{R}^d \to \mathbb{R}$ and $s, t \geq 0$, $t \leq s$,

$$E[f(X_s)|\mathcal{F}_t] = E[f(X_s)|X_t], \quad P-a.s.$$

The following theorem shows that the solution of (2.12) enjoys the Markov property.

Theorem 2.4.6 (Markov property)**.** *Let $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W)$ be an ℓ-dimensional Brownian stochastic basis and let $(X^{t,x})_{(t,x)\in[0,T]\times\mathbb{R}^d}$ be the solution of (2.12) whose coefficients satisfy the conditions of Assumption 2.4.3. Then $(X^{t,x})_{(t,x)\in[0,T]\times\mathbb{R}^d}$ fulfills the Markov property.*

Proof. For a proof we refer to Mao [2007], Theorem 2.9.1.

□

For later use, we will give here some information about the differentiability of SDEs with respect to x. We first need to introduce some notation. For some function $v \in C^{1,2}([0,T]\times\mathbb{R}^d)$, v_t denotes the partial derivative of v with respect to t and $\nabla_x v$ denotes the Jacobian matrix of the first order partial derivatives of v with respect to x.
Now, we need the following assumption on the coefficients of the stochastic differential equation.

Assumption 2.4.7. *Let σ^j, with $1 \leq j \leq \ell$, denote the j-th column of matrix σ. Let $b \in C_b^{0,1}([0,T]\times\mathbb{R}^d;\mathbb{R}^d)$ and $\sigma^j \in C_b^{0,1}([0,T]\times\mathbb{R}^d;\mathbb{R}^d)$ with $1 \leq j \leq \ell$ such that the partial derivatives $\nabla_x b$ and $\nabla_x \sigma^j$ are uniformly bounded.*

Theorem 2.4.8 (Classical differentiability)**.** *Let Assumptions 2.4.3 and 2.4.7 hold. Then the family of solution processes $(X^{t,x})_{(t,x)\in[0,T]\times\mathbb{R}^d}$ of (2.12) is continuously differentiable. Let the $d\times d$ Jacobian matrix of $(X^{t,x})_{(t,x)\in[0,T]\times\mathbb{R}^d}$ be denoted by $(\nabla_x X^{t,x})_{(t,x)\in[0,T]\times\mathbb{R}^d}$. Then, for $0 \leq t \leq s \leq T$,*

$$\nabla_x X_s^{t,x} = I_d + \int_t^s \nabla_x b(\tau, X_\tau^{t,x})\nabla_x X_\tau^{t,x} d\tau + \int_t^s \sum_{j=1}^{\ell} \nabla_x \sigma^j(\tau, X_\tau^{t,x})\nabla_x X_\tau^{t,x} dW_\tau^j. \quad (2.16)$$

Furthermore, for any $p \geq 1$, there exists a constant $C > 0$ such that for any $(t,x) \in [0,T]\times\mathbb{R}^d$

$$E\left[\sup_{s\in[t,T]} \left\|\nabla_x X_s^{t,x}\right\|^p\right] \leq C. \quad (2.17)$$

Proof. For a proof see Stroock [1982], Theorem 3.3. Note that in Stroock [1982], inequality (2.17) is shown for $p \geq 2$. The result can be extended to $p \in [1,2)$ by using Hölder's inequality.

□

2.4.2. Weak Solutions

We saw that the strong solution of a stochastic differential equation is based on a given Brownian stochastic basis. In contrast, the concept of a weak solution does not depend on a prescribed Brownian basis. The latter is part of the solution concept.

Definition 2.4.9. The 6-tuple $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W, (X^{t,x})_{(t,x)\in[0,T]\times\mathbb{R}^d})$ is called *weak solution* of (2.12) if for $(t,x) \in [0,T] \times \mathbb{R}^d$ the following properties hold

(i) $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W)$ is an ℓ-dimensional Brownian stochastic basis,

(ii) $X^{t,x} = (X_s^{t,x})_{s\in[t,T]}$ is $(\mathcal{F}_s)_{s\geq 0}$-adapted and continuous,

(iii) conditions (ii) to (iv) of Definition 2.4.1 hold.

It is clear from the definitions that strong solvability implies weak solvability. There are two concepts of uniqueness which can be associated with weak solutions.

Definition 2.4.10. We say that *pathwise uniqueness* holds for equation (2.12) whenever two weak solutions $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W, (X^{t,x})_{(t,x)\in[0,T]\times\mathbb{R}^d})$ and $(\Omega, \mathcal{F}, \left(\tilde{\mathcal{F}}_s\right)_{s\geq 0}, P, W, (\tilde{X}^{t,x})_{(t,x)\in[0,T]\times\mathbb{R}^d})$ on a common probability space $(\Omega, \mathcal{F}, P)$ with a common Brownian motion W (relative to possibly different filtrations $(\mathcal{F}_s)_{s\geq 0}$ and $\left(\tilde{\mathcal{F}}_s\right)_{s\geq 0}$) satisfy $P(X_s^{t,x} = \tilde{X}_s^{t,x}, s \in [t,T]) = 1$ for all $(t,x) \in [0,T] \times \mathbb{R}^d$.

Definition 2.4.11. We say that *uniqueness in law* holds for equation (2.12) whenever two weak solutions $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W, (X^{t,x})_{(t,x)\in[0,T]\times\mathbb{R}^d})$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \left(\tilde{\mathcal{F}}_s\right)_{s\geq 0}, \tilde{P}, \tilde{W}, (\tilde{X}^{t,x})_{(t,x)\in[0,T]\times\mathbb{R}^d})$ with the same initial distribution have the same law, that is $P_{X^{t,x}} = \tilde{P}_{\tilde{X}^{t,x}}$ for all $(t,x) \in [0,T] \times \mathbb{R}^d$.

The following result shows that the concept of pathwise uniqueness is stronger than that of uniqueness in law.

Theorem 2.4.12. *Pathwise uniqueness implies uniqueness in law.*

Proof. See Karatzas and Shreve [1991], Proposition 5.3.20, page 309.

□

We end this section with a last important connection between strong and weak solutions of SDEs.

Theorem 2.4.13. *The existence of a weak solution and pathwise uniqueness imply the existence of a strong solution.*

Proof. See Karatzas and Shreve [1991], Corollary 5.3.23.

□

2.5. (Forward-)Backward Stochastic Differential Equations

2.5.1. Backward Stochastic Differential Equations

Backward stochastic differential equations (in short BSDEs) were first introduced by Bismut [1973] to account for adjoint processes in the stochastic version of Pontryagin's maximum principle which we will introduce in the next chapter. Pardoux and Peng [1990] generalized the notion and were the first to consider general BSDEs and study the question of existence and uniqueness.
As before, we assume that $T > 0$ be a finite time horizon, and consider the given ℓ-dimensional Brownian stochastic basis $\nu = (\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W)$. We will need the following assumption.

Assumption 2.5.1. *For $m, \ell \in \mathbb{N}$, let ξ be an $\mathbb{R}^m$-valued $\mathcal{F}_T$-measurable random variable, and let $f : \Omega \times [0,T] \times \mathbb{R}^m \times \mathbb{R}^{m\times\ell} \to \mathbb{R}^m$ be a random map such that for all $(y,z) \in \mathbb{R}^m \times \mathbb{R}^{m\times\ell}$ the process $f(t,\cdot,\cdot)$ is $\mathcal{F}_t$-adapted for every $t \in [0,T]$.*

We are interested in solving the BSDE

$$\begin{cases} dY_t = -f(t, Y_t, Z_t)dt + Z_t dW_t, \quad t \in [0,T], \\ Y_T = \xi, \end{cases} \tag{2.18}$$

or equivalently,

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s.$$

The function f is called *driver* of the BSDE and ξ *terminal condition.* Note that in contrast to (forward) stochastic differential equations which were introduced in the previous section, we do not have a given *starting* value of the differential equation, but instead a given *terminal* value $Y_T = \xi$. That means that the equation is to be solved backwards in time, which justifies the name *Backward* Stochastic Differential Equation. The process Z can be interpreted as a *control process* steering the *value process* Y into the terminal condition ξ.
Let us now specify what we mean by a solution to equation (2.18).

Definition 2.5.2. A solution to BSDE (2.18) with terminal condition and generator that fulfill Assumption 2.5.1 is a pair of processes $(Y_t, Z_t)_{t\in[0,T]}$ such that

1. Y and Z are progressively measurable and respectively $\mathbb{R}^m$- and $\mathbb{R}^{m\times\ell}$-valued,
2. $\int_0^T (|f(s, Y_s, Z_s)| + \|Z_s\|^2)ds < \infty$ P-a.s.,
3. $Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - Z_s dW_s, \quad t \in [0,T]$, P-a.s.

Of course we are interested in the question of existence and uniqueness of a solution to BSDEs. As already mentioned, it was Pardoux and Peng [1990] who showed first that there exists a unique solution for generators satisfying a Lipschitz and an integrability condition.

Assumption 2.5.3. *The function f is uniformly Lipschitz continuous w.r.t. y, z uniformly in time. That means that there exists some $L > 0$ such that for all $(t, y, y', z, z') \in [0,T] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{m\times\ell} \times \mathbb{R}^{m\times\ell}$*

$$|f(t,y,z) - f(t,y',z')| \leq L(|y-y'| + \|z-z'\|).$$

In addition the following integrability condition holds

$$E\left[|\xi|^2 + \int_0^T |f(r,0,0)|^2 dr\right] < \infty.$$

Theorem 2.5.4 (Pardoux and Peng (1990))**.** *Let Assumptions 2.5.1 and 2.5.3 hold. Then the BSDE (2.18) has a unique solution $(Y,Z) \in \mathcal{S}^2_\nu([0,T];\mathbb{R}^m) \times \mathcal{H}^2_\nu([0,T];\mathbb{R}^{m\times\ell})$.*

Proof. For a proof see Pham [2008], Theorem 6.2.1.

□

We next state the Comparison Theorem first obtained by Peng [1992] in case $m = 1$.

Theorem 2.5.5. *Let (ξ^1, f^1) and (ξ^2, f^2) be two pairs of terminal conditions and generators satisfying Assumptions 2.5.1 and 2.5.3, and let $(Y^1_t, Z^1_t)_{t\in[0,T]}$, $(Y^2_t, Z^2_t)_{t\in[0,T]}$ be the solutions to their corresponding BSDEs. Suppose that*

- $P(\xi^1 \leq \xi^2) = 1$,
- $P(f^1(t, Y^1_t, Z^1_t) \leq f^2(t, Y^1_t, Z^1_t)$ *a.e.* $t \in [0,T]) = 1$.

Then $Y^1_t \leq Y^2_t$ for all $0 \leq t \leq T$, a.s.

Proof. For a proof we refer to Pham [2008], Theorem 6.2.2.

□

2.5.2. Markovian Backward Stochastic Differential Equations

In the following, we will consider *Markovian BSDEs*. This particular framework corresponds to the case where the backward equation contains some additional randomness since the solution of some forward SDE enters the driver and the terminal condition of the BSDE.
We still consider an ℓ-dimensional Brownian stochastic basis $\nu = (\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W)$. Let $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times\ell}$ be two continuous functions and $T > 0$ be a fixed final time. We assume that these functions satisfy Assumption 2.4.3, i.e., we assume that there exists a positive constant K such that for all $s \in [0,T]$ and all $x, x' \in \mathbb{R}^d$ we have

$$|b(s,x) - b(s,x')| + \|\sigma(s,x) - \sigma(s,x')\| \leq K|x - x'|, \quad (2.19)$$

$$|b(s,x)| + \|\sigma(s,x)\| \leq K(1 + |x|). \quad (2.20)$$

Then Theorem 2.4.4 gives that for $t \in [0,T]$ and any $x \in \mathbb{R}^d$ the SDE

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dW_r, \quad s \in [t,T], \tag{2.21}$$

possesses a unique strong solution.
Let us now consider two functions $k : \mathbb{R}^d \to \mathbb{R}^m$ and $f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m\times\ell} \to \mathbb{R}^m$. We assume that these two functions satisfy the following condition.

Assumption 2.5.6. *There exist two real constants $K > 0$ and $p > 0$ such that for all $(s,x,y,y',z,z') \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{m\times\ell} \times \mathbb{R}^{m\times\ell}$*

$$|f(s,x,y,z) - f(s,x,y',z')| \leq K(|y-y'| + \|z-z'\|),$$

and

$$|k(x)| + |f(s,x,y,z)| \leq K(1+|x|^p).$$

Similarly as for BSDEs which are not influenced by the solution of some SDE, one can show that for coefficients that fulfill inequalities (2.19) and (2.20) and Assumption 2.5.6 the BSDE

$$Y_s^{t,x} = k(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x}dW_r, \quad s \in [t,T], \tag{2.22}$$

has a unique solution for any $(t,x) \in [0,T] \times \mathbb{R}^d$.

Proposition 2.5.7. *Let inequalities (2.19), (2.20) and Assumption 2.5.6 hold. Then for any $(t,x) \in [0,T]\times\mathbb{R}^d$ equation (2.22) has a unique solution $(Y^{t,x}, Z^{t,x}) \in \mathcal{S}_\nu^2([t,T];\mathbb{R}^m)\times \mathcal{H}_\nu^2([t,T];\mathbb{R}^{m\times\ell})$.*
Furthermore, there exists a constant $C > 0$ such that for all $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$E\left[\sup_{s\in[t,T]} |Y_s^{t,x}|^2 + \int_t^T \left\|Z_r^{t,x}\right\|^2 dr\right] \leq C(1+|x|^2).$$

Suppose that f and k are globally Lipschitz with respect to x (uniformly in s for f). Then there exists a constant $C > 0$ such that for all $(t,x), (t',x') \in [0,T] \times \mathbb{R}^d$

$$E\left[\sup_{s\in[t,T]} |Y_s^{t,x} - Y_s^{t',x'}|^2\right] + E\left[\int_t^T \left\|Z_s^{t,x} - Z_s^{t',x'}\right\|^2 ds\right]$$
$$\leq C\left(|x-x'|^2 + (1+|x|+|x'|)^2|t-t'|\right).$$

Proof. Under the given assumptions, the forward equation is uniquely solvable. The unique solvability of the Markovian BSDE (2.22) can then be shown similary as for non-Markovian BSDEs (see Theorem 2.5.4).
Standard a priori estimates for BSDEs together with the growth condition in Assumption

2.5.6 give the existence of a constant $C > 0$ such that for all $(t, x) \in [0, T] \times \mathbb{R}^d$

$$E\left[\sup_{s\in[t,T]} |Y_s^{t,x}|^2 + \int_t^T \left\|Z_r^{t,x}\right\|^2 dr\right] \leq C\left(1 + E\left[\sup_{s\in[t,T]} |X_s^{t,x}|^2\right]\right).$$

Now, the first inequality follows by using Theorem 2.4.4.
For the second inequality note that applying Itô's formula to $|Y_s^{t,x} - Y_s^{t',x'}|^2$ for any $s \in [t, T]$ and $(t, x), (t', x') \in [0, T] \times \mathbb{R}^d$, and using standard estimates yields the existence of a constant $C > 0$ such that

$$\begin{aligned}
& E\left[\sup_{s\in[t,T]} |Y_s^{t,x} - Y_s^{t',x'}|^2\right] + E\left[\int_t^T \left\|Z_s^{t,x} - Z_s^{t',x'}\right\|^2 ds\right] \\
\leq\ & C\Big(E[|k(X_T^{t,x}) - k(X_T^{t',x'})|^2] \\
+\ & E[\int_t^T |f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) - f(r, X_r^{t',x'}, Y_r^{t,x}, Z_r^{t,x})|^2 dr]\Big).
\end{aligned}$$

Finally, the desired result follows by the Lipschitz continuity of functions k and f and by using Theorem 2.4.4 again.
For more details see El Karoui et al. [1997], Proposition 4.1.

□

For later use, we will need information about the derivative of a Markovian BSDE with respect to the forward SDE.

Proposition 2.5.8. *Let inequalities (2.19), (2.20) and Assumption 2.5.6 hold and assume that the functions b, σ, f and k are twice continuously differentiable with respect to x with uniformly bounded derivatives. Then for each $t \in [0, T]$ the function $x \mapsto (Y_s^{t,x}, Z_s^{t,x})_{s\in[t,T]}$, where $(Y_s^{t,x}, Z_s^{t,x})_{s\in[t,T]}$ is the solution to equation (2.22), is differentiable. Let the matrices of first-order partial derivatives of $Y^{t,x}$ and $Z^{j,t,x}$, $1 \leq j \leq \ell$, with respect to x be denoted by the $m \times d$ matrix $\nabla_x Y^{t,x}$ and by the $m \times d$ matrix $\nabla_x Z^{j,t,x}$ respectively (where $Z^{j,t,x}$ is the j-th column of the matrix $Z^{t,x}$). Then, for $s \in [t, T]$*

$$\begin{aligned}
\nabla_x Y_s^{t,x} &= \nabla_x k(X_T^{t,x})\nabla_x X_T^{t,x} - \int_s^T \sum_{j=1}^{\ell} \nabla_x Z_\tau^{j,t,x} dW_\tau^j \\
&+ \left(\int_s^T \nabla_x f(\tau, X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x})\nabla_x X_\tau^{t,x} + \int_s^T \nabla_y f(\tau, X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x})\nabla_x Y_\tau^{t,x}\right. \\
&+ \left.\int_s^T \nabla_z f(\tau, X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x})\nabla_x Z_\tau^{t,x}\right) d\tau.
\end{aligned}$$

In order to simplify the notation, we define for later use

$$\nabla_x Z_s^{t,x} dW_s \equiv \sum_{j=1}^{\ell} \nabla_x Z_s^{j,t,x} dW_s^j, \quad s \in [t, T].$$

Proof. See again El Karoui et al. [1997], Proposition 4.1.

□

Remark 2.5.9. Note that when taking the conditional expectation in (2.22), we obtain an alternative representation of $(Y^{t,x})_{(t,x)\in[0,T]\times\mathbb{R}^d}$ that looks as follows for any $s \in [t,T]$.

$$Y_s^{t,x} = E\left[\int_s^T f(r, X_s^{t,x}, Y_s^{t,x}, Z_r^{t,x})dr + k(X_T^{t,x})|\mathcal{F}_s\right]. \tag{2.23}$$

We will need this representation later in Chapter 5.

Also, it is possible to show that for all $(t,x) \in [0,T]\times\mathbb{R}^d$, $Y_t^{t,x}$ is deterministic (see for example El Karoui et al. [1997], Proposition 4.2). Therefore we can define a measurable function u such that for all $(t,x) \in [0,T]\times\mathbb{R}^d$, $u(t,x) \equiv Y_t^{t,x}$. One can prove that the Markov property for SDEs transfers to BSDEs of type (2.22) in the sense that $Y_s^{t,x}$, $s \in [t,T]$, depends on s and $X_s^{t,x}$, i.e., for $(t,x) \in [0,T]\times\mathbb{R}^d$ and $s \in [t,T]$,

$$Y_s^{t,x} = u(s, X_s^{t,x}), \quad P-a.s. \tag{2.24}$$

For more details see El Karoui et al. [1997], Theorem 4.1.
The property just discussed is crucial for the correspondence between BSDEs and partial differential equations (PDEs), which we will introduce shortly in the following section.

2.5.3. Feynman Kac formula

In this section we give a short review of the relation between Markovian BSDEs and PDEs. We will first present the so-called Feynman-Kac formula which shows that the solution of a quasilinear parabolic PDE can be written in form of a BSDE. Then we show that, conversely, under smoothness conditions the function $u(t,x) = Y_t^{t,x}$, $(t,x) \in [0,T]\times\mathbb{R}^d$, is a solution in some sense of a PDE.

Proposition 2.5.10. *Let v be a function of class $C^{1,2}([0,T]\times\mathbb{R}^d)$ and suppose that there exists a constant $C > 0$ such that, for each $(t,x) \in [0,T]\times\mathbb{R}^d$,*

$$|v(t,x)| + |\nabla_x v(t,x)\sigma(t,x)| \le C(1+|x|).$$

Also, v is supposed to be the solution of the following quasilinear parabolic partial differential equation

$$\begin{aligned} &v_t(t,x) + \mathcal{L}v(t,x) + f(t,x,v(t,x),\nabla_x v(t,x)\sigma(t,x)) = 0, \quad (t,x) \in [0,T]\times\mathbb{R}^d, \\ &v(T,x) = k(x), \end{aligned} \tag{2.25}$$

where $\mathcal{L}v(t,x)$ denotes the second-order differential operator

$$\mathcal{L}v(t,x) = \frac{1}{2}tr(\sigma\sigma^*(t,x)\nabla_{xx}v(t,x)) + b(t,x)\nabla_x v(t,x).$$

Then $v(t,x) = Y_t^{t,x}$ for $(t,x) \in [0,T] \times \mathbb{R}^d$, where $(Y_s^{t,x}, Z_s^{t,x})_{s\in[t,T]}$ is the unique solution of BSDE (2.22). For $(t,x) \in [0,T]\times\mathbb{R}^d$, $(Y_s^{t,x}, Z_s^{t,x}) = (v(s, X_s^{t,x}), \nabla_x v(t, X_s^{t,x})\sigma(t, X_s^{t,x}))$, $s \in [t,T]$.

Proof. See El Karoui et al. [1997], Proposition 4.3.

□

We next give the converse property by proving that the solution to BSDE (2.22) provides a solution to PDE (2.25). Before we are able to formally present this result, we need the definition of a viscosity solution.

Definition 2.5.11. Suppose $v \in C([0,T] \times \mathbb{R}^d)$ satisfies $v(T,x) = k(x)$, $x \in \mathbb{R}^d$. Then v is called a *viscosity subsolution* (resp. *supersolution*) of PDE (2.25) if for any $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d)$, whenever $\varphi - v$ attains a local minimum (resp. maximum) at $(t,x) \in [0,T] \times \mathbb{R}^d$, the inequality

$$-\varphi_t(t,x) - \mathcal{L}\varphi(t,x) - f(t,x,\varphi(t,x), \nabla_x\varphi(t,x)\sigma(t,x)^*) \leq 0$$

(resp.

$$-\varphi_t(t,x) - \mathcal{L}\varphi(t,x) - f(t,x,\varphi(t,x), \nabla_x\varphi(t,x)\sigma(t,x)^*) \geq 0)$$

holds.
Moreover, v is called a *viscosity solution* of PDE (2.25) if it is both a viscosity subsolution and a viscosity supersolution of PDE (2.25).

Theorem 2.5.12. *Let $t \in [0,T]$. Suppose that $m = 1$ and that f and k are uniformly continuous with respect to x and let $(Y^{t,x})_{(t,x)\in[0,T]\times\mathbb{R}^d}$ be the solution of (2.22). Then the function v defined by $v(t,x) = Y_t^{t,x}$ is a viscosity solution of PDE (2.25).*

Proof. See El Karoui et al. [1997], Theorem 4.2.

□

Later we will need the stability property of viscosity solutions.

Theorem 2.5.13 (Stability of Viscosity Solutions)**.** *Assume that $(b^n, \sigma^n, f^n, k^n)_{n\in\mathbb{N}}$ converges to (b, σ, f, k) locally uniformly as $n \to \infty$ in the respective domain of definition. Assume that for $n \in \mathbb{N}$, v^n is a viscosity solution to*

$$\begin{cases} -v_t^n - \mathcal{L}^n v^n(t,x) - f^n(t,x,v^n(t,x), \nabla_x v^n(t,x)\sigma^n(t,x)) = 0, & (t,x) \in [0,T) \times \mathbb{R}^d, \\ v^n(T,x) = k^n(x), \quad x \in \mathbb{R}^d, \end{cases}$$

where

$$\mathcal{L}^n v^n(t,x) = \frac{1}{2} tr(\sigma^n \sigma^{n*}(t,x)\nabla_{xx} v^n(t,x)) + b^n(t,x)\nabla_x v^n(t,x),$$

such that $v^n \to v$ (locally uniformly) as $n \to \infty$. Then v is a viscosity solution of

$$\begin{cases} -v_t - \mathcal{L}v(t,x) - f(t,x,v(t,x), \nabla_x v(t,x)\sigma(t,x)) = 0, & (t,x) \in [0,T) \times \mathbb{R}^d, \\ v(T,x) = k(x), \quad x \in \mathbb{R}^d. \end{cases}$$

Proof. For a proof see Bardi et al. [1997], Proposition 8.1, p. 20.

2.5.4. Forward-Backward Stochastic Differential Equations

We saw in the previous section that a Markovian BSDE defines a system of SDEs consisting of a forward SDE and a backward SDE, where the solution of the forward equation enters the backward equation but not vice versa. In this section we will give a short overview about a system of SDEs where the solution of the backward equation enters the forward equation as well. Such a system is called *Forward-Backward Stochastic Differential Equation* (for short: FBSDE).
Let in the following $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W)$ be a given ℓ-dimensional Brownian stochastic basis, $T > 0$ be a fixed deterministic final time and consider the functions $b : [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m\times\ell} \to \mathbb{R}^d$, $\sigma : [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m\times\ell} \to \mathbb{R}^{d\times\ell}$, $f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m\times\ell} \to \mathbb{R}^m$ and $k : \mathbb{R}^d \to \mathbb{R}^m$. Then a FBSDE generally takes the following form for any $(t,x) \in [0,T] \times \mathbb{R}^d$:

$$\begin{cases} dX_s = b(s, X_s, Y_s, Z_s)ds + \sigma(s, X_s, Y_s, Z_s)dW_s, \\ dY_s = f(s, X_s, Y_s, Z_s)ds + Z_s dW_s, \quad s \in [t,T], \\ X_t = x, \quad Y_T = k(X_T). \end{cases} \tag{2.26}$$

As for Markovian BSDEs, X, Y and Z are the unknown processes, and they are required to be $(\mathcal{F}_s)_{s\geq 0}$-adapted. Since all three processes (X, Y, Z) appear in each of the coefficients of equation (2.26), we call the above system a *fully coupled* FBSDE. If b and σ do not involve Y and Z, we obtain a Markovian BSDE which is sometimes also called *decoupled FBSDE*.

Definition 2.5.14. For any $(t,x) \in [0,T] \times \mathbb{R}^d$ a triple of processses $(X^{t,x}, Y^{t,x}, Z^{t,x}) \in \mathcal{S}^2_\nu([t,T]; \mathbb{R}^d) \times \mathcal{S}^2_\nu([t,T]; \mathbb{R}^m) \times \mathcal{H}^2_\nu([t,T]; \mathbb{R}^{m\times\ell})$ is called an *(adapted) solution* of the FBSDE (2.26) if the following holds:

$$\begin{cases} X^{t,x}_s = x + \int_t^s b(r, X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r)dr + \int_t^s \sigma(r, X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r)dW_r, \\ Y^{t,x}_s = k(X^{t,x}_T) - \int_t^s f(r, X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r)dr + \int_t^s Z^{t,x}_r dW_r, \quad s \in [t,T], \quad P-a.s. \end{cases}$$

Equation (2.26) is said to have a *unique* adapted solution if for any two adapted solutions $(X^{t,x}, Y^{t,x}, Z^{t,x})_{t\in[0,T]\times\mathbb{R}^d}$ and $(\tilde{X}^{t,x}, \tilde{Y}^{t,x}, \tilde{Z}^{t,x})_{t\in[0,T]\times\mathbb{R}^d}$,

$$P\left((X^{t,x}_s, Y^{t,x}_s) = (\tilde{X}^{t,x}_s, \tilde{Y}^{t,x}_s), s \in [t,T], \text{ and } Z^{t,x}_s = \tilde{Z}^{t,x}_s, \ a.e.\ s \in [t,T]\right) = 1.$$

Fully coupled FBSDEs are not necessarily solvable. We refer to Ma and Yong [1999] for examples of nonsolvable FBSDEs. There are mainly three approaches for the wellposedness of FBSDEs in the literature, each of which has its constraints. The first method tries to adopt a concept related to contraction mappings that works fine for (Markovian) BSDEs to prove the solvability of FBSDEs. With this method one has to assume that T is small enough (see Antonelli [1993] or Ma and Yong [1999], Chapter 1, §5.). The second method is the four step scheme by Ma et al. [1994]. This method allows T to be arbitrary large, but requires the coefficients to be deterministic and σ to be nonde-

generate and independent of Z. Ma et al. [1994] prove that in this case the backward components of the adapted solution are determined explicitly by the forward component via the solution of a certain quasi-linear parabolic PDE system (i.e., the solution (Y, Z) is obtained by the solution of a PDE very similarly as in Proposition 2.5.10). Finally there is the method of continuation (see for example Hu and Peng [1995] or Peng and Wu [1999]) which allows T to be large, the coefficients to be random and does not require the forward equation to be non-degenerate. However, it requires certain monotonicity conditions in order to prove the existence and uniqueness of solutions to FBSDEs.
For later use we present here some version of the existence and uniqueness result of Delarue [2002]. There a Markovian framework is considered where the diffusion coefficient of the forward equation is independent of Z and uniformly nondegenerate. In this case an FBSDE over arbitrary time duration was solved under Lipschitz conditions on the coefficients, by combining nicely the method of contraction mapping, the four step scheme and some PDE arguments.
As in Delarue [2002] let us consider a system of equations similar to (2.26) with the difference that Z does not enter σ, i.e., we have the functions $b : [0,T]\times\mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^{m\times\ell} \to \mathbb{R}^d$, $\sigma : [0,T]\times\mathbb{R}^d\times\mathbb{R}^m \to \mathbb{R}^{d\times\ell}$, $f : [0,T]\times\mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^{m\times\ell} \to \mathbb{R}^m$ and $k : \mathbb{R}^d \to \mathbb{R}^m$ which enter the following equations. For any $(t,x) \in [0,T]\times\mathbb{R}^d$,

$$\begin{cases} dX_s = b(s,X_s,Y_s,Z_s)ds + \sigma(s,X_s,Y_s)dW_s, \\ dY_s = f(s,X_s,Y_s,Z_s)ds + Z_s dW_s, \quad s \in [t,T], \\ X_t = x, \quad Y_T = k(X_T). \end{cases} \tag{2.27}$$

We impose the following assumption on the coefficients.

Assumption 2.5.15. *For the functions b, σ, f and k in (2.27) there exist three constants C, Λ and $\kappa > 0$ such that*

- *for all $(s,x,y,z) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^{m\times\ell}$*

$$|b(s,x,y,z)| + \|\sigma(s,x,y)\| + |f(s,x,y,z)| + |k(x)| \le \Lambda(1 + |y| + \|z\|);$$

- *for all $(s,x,y) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^m$ and for all $\varsigma \in \mathbb{R}^{d\times d}$,*

$$\langle \varsigma, a(s,x,y)\varsigma\rangle \ge \kappa|\varsigma|^2,$$

where $a = \sigma\sigma^*$;

- *for all $s \in [0,T]$ and $(x,y,z), (x',y',z') \in \mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^{m\times\ell}$,*

$$\begin{aligned} & |b(s,x,y,z) - b(s,x',y',z')| + \|\sigma(s,x,y) - \sigma(s,x',y')\| \\ + \quad & |f(s,x,y,z) - f(s,x',y',z')| + |k(x) - k(x')| \\ \le \quad & C\left(|x-x'| + |y-y'| + \|z-z'\|\right). \end{aligned}$$

Theorem 2.5.16. *Let Assumption 2.5.15 be satisfied. Then for all $T > 0$ and for all $(t,x) \in [0,T]\times\mathbb{R}^d$ there exists a unique solution $(X^{t,x}, Y^{t,x}, Z^{t,x})$ of FBSDE (2.27).*

Proof. For a proof see Delarue [2002], Theorem 2.6.

□

Remark 2.5.17. The problem of showing the existence and uniqueness of a solution to a FBSDE was addressed by many authors in the past years. Only a few are mentioned above. Most of the papers have in common that the continuity of the coefficients is essential. Until today there are very few works that consider discontinuous coefficients. See for example Delarue and Guatteri [2006], where a class of non-degenerate deterministic FBSDEs is considered with backward drivers that may be discontinuous in x. In this situation it is not possible to show the existence of a strong solution, but at least Delarue and Guatteri [2006] establish the existence and uniqueness of a weak solution.

3. Introduction to Stochastic Control Theory

Stochastic control is the study of stochastic differential equations which can be controlled by some decision maker (controller). The aim of the controller is to select an optimal decision in order to optimize some performance criterion. At each point in time the decisions of the controller are based on the most updated information available at this moment. Since the controlled system is dynamic, this has the consequence that the relevant decision must also change over time. Such optimization problems are called *stochastic optimal control problems*.

This chapter will give a short overview about the theory of stochastic control and is organized as follows. In Section 3.1, we will give a formal definition of controlled stochastic differential equations. Similarly as for uncontrolled SDEs, we will introduce the concepts of strong and weak solutions. Then, we will focus on the control process and will distinguish between two different types of control processes, that are strict controls and relaxed controls. In Section 3.2, we will define the performance criterion which needs to be optimized, and in Section 3.3, we present the Bellman dynamic programming approach with the theory of viscosity solutions. In Section 3.4, we will give a short review of the stochastic maximum principle.

3.1. Controlled Stochastic Differential Equations

Let $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W)$ be an ℓ-dimensional Brownian stochastic basis and let $T > 0$ be a finite time horizon. Let U be a separable metric space and let $u = (u_s)_{s\in[0,T]}$ be a given $(\mathcal{F}_s)_{s\geq 0}$-adapted control process taking values in U. Let $b : [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}^d$, $\sigma : [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}^{d\times\ell}$ and consider for $(t,x) \in [0,T] \times \mathbb{R}^d$ the following d-dimensional *controlled stochastic differential equation*.

$$dX_s = b(s, X_s, u_s)ds + \sigma(s, X_s, u_s)dW_s, \quad X_t = x \in \mathbb{R}^d, \quad s \in [t,T]. \tag{3.1}$$

The latter equation can be written equivalently as a stochastic integral equation of the form

$$X_s = x + \int_t^s b(r, X_r, u_r)dr + \int_t^s \sigma(r, X_r, u_r)dW_r, \quad s \in [t,T]. \tag{3.2}$$

Through the choice of u the controller can influence the behavior of X by modifying its drift and diffusion coefficient. At any time $s \in [t,T]$, the controller can use for his choice of u_s all information available up to this point of time, but of course he is not able to foretell what is going on afterwards due to the uncertainty of the system. This means

that the control process is assumed to be nonanticipative and therefore adapted to the underlying filtration.
Similarly as for ordinary (uncontrolled) stochastic differential equations we will distinguish in the following strong solutions and weak solutions of the controlled SDE.

3.1.1. Solution Concepts: Strong and Weak Solutions

In the case of uncontrolled SDEs we saw that we need some Lipschitz and linear growth conditions on the coefficients to ensure the existence of a unique strong solution. Since now additionally the control process enters the coefficients of our SDE, we need to strengthen the conditions on the coefficients. This can be accomplished by assuming that the usual Lipschitz and growth conditions hold uniformly over all $u \in U$.

Assumption 3.1.1. *Assume there exists a positive constant K such that for all $s \in [0,T]$, $u \in U$ and $x, y \in \mathbb{R}^d$ we have*

$$\begin{aligned} |b(s,x,u) - b(s,y,u)| + \|\sigma(s,x,u) - \sigma(s,y,u)\| &\leq K|x-y|, \\ |b(s,x,u)| + \|\sigma(s,x,u)\| &\leq K(1+|x|), \end{aligned}$$

and

$$u \mapsto b(s,x,u) \text{ and } u \mapsto \sigma(s,x,u) \text{ are continuous uniformly in } (s,x).$$

Now, we are able to present the theorem that proves the existence of a strong solution of (3.1).

Theorem 3.1.2. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W)$ be an ℓ-dimensional Brownian stochastic basis and let $u = (u_s)_{s\in[0,T]}$ be a given $(\mathcal{F}_s)_{s\geq 0}$-adapted process taking values in U. Let b and σ satisfy Assumption 3.1.1. Then there exists a unique strong solution $(X^{t,x,u})_{(t,x)\in[0,T]\times\mathbb{R}^d}$ of (3.1), i.e., for any $(t,x) \in [0,T]\times\mathbb{R}^d$, there is some $(\mathcal{F}_s)_{s\geq 0}$-adapted continuous process $X^{t,x,u} = (X_s^{t,x,u})_{s\in[t,T]}$ such that*

(i) $P(X_t^{t,x,u} = x) = 1$,

(ii) $\int_t^s \left\{ |b(r, X_r^{t,x,u}, u_r)| + \|\sigma(r, X_r^{t,x,u}, u_r)\|^2 \right\} dr < \infty$, *for all $s \in [t,T]$, P-a.s.,*

(iii) $X^{t,x,u}$ satisfies the stochastic integral equation (3.2).

Furthermore, for any $p \geq 1$, there exists a constant $C > 0$ such that for all $(t,x) \in [0,T]\times\mathbb{R}^d$

$$E\left[\sup_{s\in[t,T]} |X_s^{t,x,u}|^p\right] \leq C(1+|x|^p), \tag{3.3}$$

and for all $(t,x), (t',x') \in [0,T]\times\mathbb{R}^d$, $s \in [t,T]$ and $s' \in [t',T]$,

$$E\left[|X_s^{t,x,u} - X_s^{t',x',u}|^p\right] \leq C\left(|x-x'|^p + (1+|x|+|x'|)^p\left(|t-t'|^{p/2} + |s-s'|^{p/2}\right)\right). \tag{3.4}$$

Proof. Beside notational changes the proof is the same as in the case with no control as long as the control process is $(\mathcal{F}_s)_{s\geq 0}$-adapted and measurable.

$\square$

Note that the uniqueness of the strong solution is defined for the controlled case as in Definition 2.4.2 in the previous chapter.
Similarly as in the case of uncontrolled SDEs we will introduce in addition the concept of weak solutions for controlled SDEs.

Definition 3.1.3 (Weak Solution of Controlled SDEs)**.** A 7-tuple $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W, u, (X^{t,x,u})_{(t,x)\in[0,T]\times\mathbb{R}^d})$ is called *weak solution* of (3.1) if for $(t,x) \in [0,T] \times \mathbb{R}^d$ the following properties hold

(i) $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W)$ is an ℓ-dimensional Brownian stochastic basis,

(ii) $X^{t,x,u} = \left(X_s^{t,x,u}\right)_{s\in[t,T]}$ is $(\mathcal{F}_s)_{s\geq 0}$-adapted and continuous,

(iii) $u = (u_s)_{s\in[t,T]}$ is $(\mathcal{F}_s)_{s\geq 0}$-adapted and takes values in U,

(iv) conditions (i) to (iii) of Theorem 3.1.2 hold.

3.1.2. Control Classes: Strict Controls and Relaxed Controls

In the following we will use the term *classical* or *strict* control for the control used in (3.1). In order to show the existence of optimal stochastic control processes we will later need a relaxation of the notion of the control process u defined above to that of a *relaxed* control process.
With relaxed controls, for any $t \in [0,T]$ the controller chooses at any time $s \in [t,T]$ a *probability measure* on the control set U, rather than an element $u_s \in U$.

Definition 3.1.4. A deterministic relaxed control is a positive measure λ on the Borel σ-algebra $\mathcal{B}([0,T] \times U)$ such that for any $t \in [0,T]$

$$\lambda([t,s] \times U) = s, \quad s \in [t,T]. \tag{3.5}$$

Denote by $\Lambda(U)$ the set of all deterministic relaxed controls over U.
Since U is assumed to be a separable metric space, it follows that every element λ of $\Lambda(U)$ fulfills the disintegration property. This means that there is a derivative λ' such that $\lambda(ds,du) = \lambda'(s,du)ds$. Note that $\lambda'(s,du)$ is a progressively measurable process with value in the set of probability measures $\mathcal{P}(U)$.
The space $\Lambda(U)$ is equipped with the *weak-compact topology.* In analogy to Definition 2.1.1 in Chapter 2, this implies that a sequence of relaxed controls $(\lambda_n)_{n\in\mathbb{N}}$ converges (weakly) to a relaxed control $\lambda \in \Lambda(U)$ if and only if for any $t \in [0,T]$ and any bounded and continuous function γ with compact support (i.e., $\gamma \in C_c([t,T] \times U)$)

$$\int_t^T \int_U \gamma(s,u)\lambda_n(ds,du) \overset{n\to\infty}{\longrightarrow} \int_t^T \int_U \gamma(s,u)\lambda(ds,du). \tag{3.6}$$

Remark 3.1.5. If U is assumed to be compact, it follows from Corollary 2.1.4 and Theorem 2.1.3 that the space of relaxed controls is compact. Consequently, every sequence in $\Lambda(U)$ contains a weakly convergent subsequence that converges to an element in $\Lambda(U)$.

Next, we introduce a suitable filtration on $\Lambda(U)$ in the same way as it is done in Yong and Zhou [1999]. Note first that each $\lambda \in \Lambda(U)$ can be related to a linear functional on $C([0,T] \times U)$ in the following way.

$$\lambda(f) \equiv \int_0^T \int_U f(s,u)\lambda(ds,du), \quad f \in C([0,T] \times U).$$

For any $f \in C([0,T] \times U)$ and $t \in [0,T]$ define $f^t \in C([0,T] \times U)$ by

$$f^t(s,u) \equiv f(s \wedge t, u).$$

Since $C([0,T] \times U)$ is separable and therefore has a countable dense subset, we may let $(f_j)_{j\geq 1}$ be such a countable dense subset (with respect to the supremum norm). It is easy to see that for any $t \in [0,T]$, $\left(f_j^t\right)_{j\geq 1}$ is dense in the set $\{f^t | f \in C([0,T] \times U)\}$. Define for $t \in [0,T]$

$$\mathcal{B}_t(\Lambda) \equiv \sigma\left(\{\lambda \in \Lambda(U) | \lambda(f^s) \in B\} : s \in [0,t], B \in \mathcal{B}(\mathbb{R})\right). \tag{3.7}$$

One can easily show that $\mathcal{B}_t(\Lambda)$ can be generated by cylinders of the form

$$\mathcal{B}_t(\Lambda) = \sigma\left(\left\{\lambda \in \Lambda(U) | \lambda(f_j^s) \in (a,b)\right\} : s \leq t \in \mathbb{Q}, j \geq 1 \text{ and } a,b \in \mathbb{Q}\right). \tag{3.8}$$

Definition 3.1.6. A *relaxed control process* over a compact metric space U is a $\Lambda(U)$-valued random variable λ defined on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P\right)$ such that the mapping

$$\Omega \ni \omega \mapsto \lambda([t,s] \times G)(\omega)$$

is $\mathcal{F}_s$-measurable for all $s \geq t$, $G \in \mathcal{B}(U)$.

Using the relaxed control process $\lambda \in \Lambda(U)$, resp. the corresponding derivative λ', a controlled state process on an ℓ-dimensional Brownian stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W)$ takes the following form for any initial condition $(t,x) \in [0,T] \times \mathbb{R}^d$:

$$\begin{cases} dX_s^{t,x,\lambda} = \int_U b(s, X_s^{t,x,\lambda}, u)\lambda'(s,du)ds + \int_U \sigma(s, X_s^{t,x,\lambda}, u)\lambda'(s,du)dW_s, & s \in [t,T], \\ X_t^{t,x,\lambda} = x, \end{cases} \tag{3.9}$$

where as in equation (3.1) $b : [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}^{d\times\ell}$. In order to simplify the notation, we introduce the following definition.

Definition 3.1.7. For $l(s,x,u) = b(s,x,u), \sigma(s,x,u)$ with $(s,x,u) \in [0,T] \times \mathbb{R}^d \times U$ let

$$\tilde{l}(s,x,\lambda) \equiv \int_U l(s,x,u)\lambda'(s,du).$$

Note that $\tilde{l}$ is continuous on $[0,T] \times \mathbb{R}^d \times \Lambda(U)$ and is linear in λ. With this notation, for any $(t,x) \in [0,T] \times \mathbb{R}^d$, equation (3.9) can be written as

$$\begin{cases} dX_s^{t,x,\lambda} = \tilde{b}(s, X_s^{t,x,\lambda}, \lambda)ds + \tilde{\sigma}(s, X_s^{t,x,\lambda}, \lambda)dW_s, & s \in [t,T], \\ X_t^{t,x,\lambda} = x. \end{cases} \tag{3.10}$$

If the coefficients fulfill Assumption 2.4.3, one can show similarly as with a strict control that a strong solution to the latter SDE exists. In the following, we denote the solution to (3.10) for a given relaxed control $\lambda \in \Lambda(U)$ and for any $(t,x) \in [0,T] \times \mathbb{R}^d$ as $X^{t,x,\lambda} \equiv \left(X_s^{t,x,\lambda}\right)_{s\in[t,T]}$.

Remark 3.1.8. We can embed the set of measurable functions $\Psi : \mathbb{R}_+ \to U$ into $\Lambda(U)$ by the formula $\lambda(ds,du) = \delta_{\Psi_s}(du)ds = \delta'_{\Psi_s}(s,du)ds$, where δ_z is the Dirac measure at a point z in U. Applying this notation to a strict control process u with values in U yields for any uniformly continuous function $\varphi(s,x,u_s)$ defined on $[0,T] \times \mathbb{R}^d \times U$,

$$\varphi(s,x,u_s) = \int_U \varphi(s,x,u)\delta'_{u_s}(s,du), \quad (s,x,u) \in [0,T] \times \mathbb{R}^d \times U.$$

The last remark shows that any strict control can be written as a relaxed control. The converse is not true in general. However, the following lemma, known as the *Chattering Lemma*, tells us that any relaxed control is a weak limit of a sequence of strict controls.

Lemma 3.1.9 (Chattering Lemma)**.** *Let λ be a predictable process with values in the space of probability measures on U. Then there exists a sequence of predictable processes $(u^n)_{n\in\mathbb{N}}$ with values in U such that the sequence of random measures $(\delta_{u_s^n}(du)ds)$ converges weakly to $\lambda(ds,du)$, P-a.s.*

Proof. This lemma was originally introduced for deterministic measures in Ghouila-Houri [1967] and extended to random measures in Fleming [1977] and El Karoui et al. [1988].

□

We say that λ is an $(\mathcal{F}_s)_{s\geq 0}$-adapted $\Lambda(U)$-valued random variable on a given filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P\right)$ if $\lambda(\cdot, B)$ is $(\mathcal{F}_s)_{s\geq 0}$-measurable for any $B \in \mathcal{B}([0,T] \times U)$. From the last remark, it is clear that if u is an $(\mathcal{F}_s)_{s\geq 0}$-adapted U-valued process, then its embedding $\delta_{u.}$ is an $(\mathcal{F}_s)_{s\geq 0}$-adapted $\Lambda(U)$-valued random variable and vice versa.

3.2. The Cost Functional and Admissible Controls

In the previous section in equation (3.1), we have introduced controlled stochastic differential equations. In the framework of stochastic optimal control, this controlled SDE is called the *state process*. A second component in the treatment of stochastic optimal control problems is the so called *cost functional*, which in general takes the form

$$J(t,x,u) = E\left[\int_t^T f(s, X_s^{t,x,u}, u_s)ds + k(X_T^{t,x,u})\right], \tag{3.11}$$

where $x \in \mathbb{R}^d$ denotes the starting value of the state process (3.1) starting at time $t \in [0,T]$, $f : [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}$ (the *running cost*), and $k : \mathbb{R}^d \to \mathbb{R}$ (the *terminal cost*).

Besides Assumption 2.4.3 which ensures the existence of a strong solution of the state process (3.1), we need some assumptions on the functions f and k in the cost functional in order to ensure that it is well defined.

Assumption 3.2.1. *Assume there exists a positive constant K such that for all $s \in [0,T]$, $u \in U$ and $x, y \in \mathbb{R}^d$ we have*

$$\begin{aligned} |f(s,x,u) - f(s,y,u)| + |k(x) - k(y)| &\leq K|x-y|, \\ |f(s,x,u)| + |k(x)| &\leq K(1+|x|), \end{aligned}$$

and

$$u \mapsto f(s,x,u) \text{ is continuous uniformly in } (s,x).$$

We see that the control variable u enters the state process as well as the cost functional. By choice of $u_s \in U$ for any $s \in [t,T]$, the controller can influence the cost functional in two ways: on the one hand by the choice of u directly since u enters the function f, and on the other hand indirectly by his influence on the solution of the state process $X^{t,x,u}$ which also enters the cost functional. The aim of the controller is to choose an admissible strategy such that the expected costs described by the cost functional, are minimized. As already mentioned, we need the control process to be nonanticipative (adapted) in order to be admissible. In the strong setting, the filtration is fixed and given. In the weak setting however, the filtration is part of the solution and may vary. Therefore, the following definition of admissibility of control processes depends on the choice of the Brownian stochastic basis.

Definition 3.2.2 (Admissible Controls)**.** A 6-tuple $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W, u)$ is called an *admissible control*, and $(X^{t,x,u}, u)_{(t,x)\in[0,T]\times\mathbb{R}^d}$ an *admissible pair*, if for all $(t,x) \in [0,T] \times \mathbb{R}^d$

- $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W)$ is an ℓ-dimensional Brownian stochastic basis,
- u is an $(\mathcal{F}_s)_{s\geq 0}$-adapted process on $(\Omega, \mathcal{F}, P)$ taking values in U,
- $X^{t,x,u}$ is the unique solution of (3.1) under u in the sense of Theorem 3.1.2,
- the control process u is such that

$$E\left[\int_t^T |f(s, X_s^{t,x,u}, u_s)|ds + |k(X_T^{t,x,u})|\right] < \infty.$$

Let us denote in the following the Brownian stochastic basis which is part of the considered 6-tuple as $\nu = (\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W)$. Then on this stochastic basis, $\mathcal{U}_\nu[t,T]$, $t \in [0,T]$, denotes the set of *admissible controls.*

The optimal control problem can be stated as follows.

Problem (P): Minimize (3.11) subject to (3.1) over $\mathcal{U}_\nu[t,T]$, $t \in [0,T]$.

Definition 3.2.3. For any $(t,x) \in [0,T] \times \mathbb{R}^d$, we call $(X^{t,x,\bar{u}}, \bar{u})$ an *optimal pair* if the trajectory $X^{t,x,\bar{u}}$ given by the control $\bar{u} \in \mathcal{U}_\nu[t,T]$ solves Problem (P), i.e.,

$$J(t,x,\bar{u}) = \inf_{u \in \mathcal{U}_\nu[t,T]} J(t,x,u). \tag{3.12}$$

$\bar{u}$ is called an *optimal control* and the corresponding process $\bar{X}^{t,x} \equiv X^{t,x,\bar{u}}$ is called an *optimal state process (trajectory).*

Definition 3.2.4. An admissible strategy u is called a *Markov strategy* or a *feedback control* if for any $(t,x) \in [0,T] \times \mathbb{R}^d$ it is of the form $u_s = v(s, X_s^{t,x,u})$, $s \in [t,T]$, for some measurable function $v : [t,T] \times \mathbb{R}^d \to U$.

The reason for this terminology is the fact that for a Markov strategy, the controlled SDE is a Markov process. This is in general not true, since the control u may depend on the entire past and is only required to be adapted.
So far only strict controls u taking values in U were considered for the cost functional. It turns out that in many situations an optimal control does not necessarily exist in U. For an example see Kushner and Dupuis [2001], p. 86. The reason for this is that the set U is not equipped with a compact topology. And for proving the existence of optimal controls, compactness is very useful. In this situation relaxed controls which were defined in the previous section are helpful. As explained in Remark 3.1.5, the space of relaxed controls enjoys the desired compactness property if U is compact.
In (3.9), resp. (3.10), we describe the state process in case of relaxed controls. Similarly, we can extend the definition of a cost functional to the case of relaxed controls, by writing

$$J(t,x,\lambda) = E\left[\int_t^T \tilde{f}(s, X_s^{t,x,\lambda}, \lambda) ds + k(X_T^{t,x,\lambda})\right], \quad (t,x) \in [0,T] \times \mathbb{R}^d, \tag{3.13}$$

where $\lambda \in \Lambda(U)$, $X^{t,x,\lambda}$ denotes the solution of the SDE (3.9) for any $(t,x) \in [0,T] \times \mathbb{R}^d$, and $\tilde{f}(s,x,\lambda) = \int_U f(s,x,u)\lambda'(s,du)$ for any $(s,x,u) \in [0,T] \times \mathbb{R}^d \times U$. Recall that $\Lambda(U)$ is the set of all deterministic relaxed controls over U (compare Definition 3.5).
Similarly as in the case with strict controls, we can define admissibility.

Definition 3.2.5 (Admissible Relaxed Controls)**.** A 6-tuple $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W, \lambda)$ is called an *admissible relaxed control*, and $(X^{t,x,\lambda}, \lambda)_{(t,x)\in[0,T]\times\mathbb{R}^d}$ an admissible pair, if for all $(t,x) \in [0,T] \times \mathbb{R}^d$,

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{s\geq 0}, P, W)$ is an ℓ-dimensional Brownian stochastic basis,
- λ with values in $\Lambda(U)$ is a relaxed control defined on the Brownian stochastic basis,
- $X^{t,x,\lambda}$ is the unique solution of (3.10) under λ,
- the control process λ is such that

$$E\left[\int_t^T \int_U |f(s, X_s^{t,x,\lambda}, u)|\lambda'(s, du)ds + |k(X_T^{t,x,\lambda})|\right] < \infty.$$

On the stochastic basis $\nu = (\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W)$, the set of admissible relaxed controls is denoted as $\hat{\mathcal{U}}_\nu[t, T]$, $t \in [0, T]$.

Similarly as for the strict control problem, we can formulate the aim of the relaxed control problem as follows.

Problem (RP): Minimize (3.13) subject to (3.10) over $\hat{\mathcal{U}}_\nu[t, T]$, $t \in [0, T]$.

Between Problem (P) and Problem (RP) the following link exists.
In contrast to the space U, the space $\Lambda(U)$ is equipped with a weak-compact topology (see (3.6)). By using compactification techniques, one can therefore prove the existence of an optimal relaxed control. It was Fleming [1977] who derived the first existence result of an optimal relaxed control for SDEs with uncontrolled diffusion coefficient under the assumption that the coefficients are Lipschitz continuous. The case of an SDE where the diffusion coefficient depends explicitly on the control variable has been studied by El Karoui et al. [1987], where the optimal relaxed control is shown to be Markovian.
Under some convexity assumption on the coefficients of the control problem (the so-called *Roxin condition* which is given in detail in Chapter 5), it is possible to prove that for each admissible relaxed control $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W, \lambda)$, there exists a U-valued $(\mathcal{F}_s)_{s\geq 0}$-adapted process u such that $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W, u)$ is a control for which the cost functional takes the same value (see for example Theorem 2.10 in El Karoui et al. [1987]). Consequently, for the existing optimal relaxed control, there is some strict control that admits the same expected cost as the relaxed control.
This means that the existence of a (optimal) relaxed control to problem (RP) ensures the existence of an admissible strict control. Now, by a result of Kushner [1975] (compare Theorem 3.1 in Kushner [1975]), one can show that the convexity assumption on the coefficients and the fact that there exists *some* admissible strict control u ensure the existence of an *optimal* admissible strict control, i.e., a solution to Problem (P).

3.3. The Value Function, Hamilton-Jacobi-Bellman Equation and Viscosity Solutions

In this section we will introduce the value function of a stochastic control problem which is defined to be the infimum of the cost functional as a function of the initial data. We will see that if the value function is smooth enough, it solves a partial differential equation (PDE) which we call the Hamilton-Jacobi-Bellman Equation. Since unfortunately the smoothness of the value function is more of an exception than a rule, it is necessary to introduce the concept of viscosity solutions which describe a weak formulation of solutions of this kind of PDEs.

In the following we will consider for simplicity only the strong formulation of the stochastic control problem based on strict controls. But it should be mentioned that the following definitions and results can be transferred also to the weak setting or to relaxed controls.

We assume that an ℓ-dimensional Brownian basis $\nu = (\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W)$ is given and let $u \in \mathcal{U}_\nu[t,T]$ be an admissible control taking values in U. For $(t,x) \in [0,T] \times \mathbb{R}^d$, we consider the following state process

$$dX_s^{t,x,u} = b(s, X_s^{t,x,u}, u_s)ds + \sigma(s, X_s^{t,x,u}, u_s)dW_s, \quad X_t^{t,x,u} = x, \quad s \in [t,T]. \tag{3.14}$$

The solution of the state process $(X^{t,x,u})_{(t,x)\in[0,T]\times\mathbb{R}^d}$ enters the cost functional which takes the form

$$J(t,x,u) = E\left[\int_t^T f(s, X_s^{t,x,u}, u_s)ds + k(X_T^{t,x,u})\right], \quad (t,x) \in [0,T] \times \mathbb{R}^d, u \in \mathcal{U}_\nu[t,T]. \tag{3.15}$$

The aim is to determine the optimal control in the set $\mathcal{U}_\nu[t,T]$ that minimizes the cost functional J. We therefore define the following function which is called the *value function* of Problem (P):

$$\begin{aligned} &V(t,x) = \inf_{u\in\mathcal{U}_\nu[t,T]} J(t,x,u), \quad (t,x) \in [0,T] \times \mathbb{R}^d, \\ &V(T,x) = k(x), \quad x \in \mathbb{R}^d. \end{aligned} \tag{3.16}$$

An important tool in the theory of stochastic control is the *dynamic programming principle* which is the stochastic version of Bellman's principle of optimality. Bellman describes the idea behind this principle of optimality as follows: "An optimal policy has the property that, whatever the initial state and initial decision are, the remaining decision must consitute an optimal policy with regard to the outcome resulting from the first decision." Mathematically, this statement can be formulated as follows.

Theorem 3.3.1. *Let Assumptions 3.1.1 and 3.2.1 hold. Then for any $(t,x) \in [0,T] \times \mathbb{R}^d$*

$$V(t,x) = \inf_{u\in\mathcal{U}_\nu[t,T]} E\left[\int_t^{\hat{t}} f(s, X_s^{t,x,u}, u_s)ds + V(\hat{t}, X_{\hat{t}}^{t,x,u})\right], \quad 0 \leq t \leq \hat{t} \leq T. \tag{3.17}$$

Proof. See Yong and Zhou [1999], Chapter 4, Theorem 3.3.

□

The advantage of this method is that it is not necessary to optimize the control strategy u over the entire time interval $[t,T]$ at once. One can divide the time interval into smaller chunks and optimize on each one individually.
From the last result, one can deduce the following.

Theorem 3.3.2. *Let Assumptions 3.1.1 and 3.2.1 hold. If $(\bar{X}^{t,x}, \bar{u})_{(t,x)\in[0,T]\times\mathbb{R}^d}$ is optimal for Problem (P), then*

$$V(s, \bar{X}_s^{t,x}) = E\left[\int_s^T f(r, \bar{X}_r^{t,x}, \bar{u}_r)dr + k(\bar{X}_T^{t,x})|\mathcal{F}_s\right], \quad P-a.s., s \in [t,T]. \tag{3.18}$$

Proof. See Yong and Zhou [1999], Chapter 4, Theorem 3.4.

□

The infinitesimal version of the dynamic programming principle is the so called *Hamilton-Jacobi-Bellman equation* (HJB equation for short). The HJB equation describes a partial differential equation that the value function V should satisfy.

Proposition 3.3.3. *Suppose Assumptions 3.1.1 and 3.2.1 hold and $V \in C^{1,2}([0,T]\times\mathbb{R}^d)$. Then V is a solution of the following terminal value problem of the (possibly degenerate) second-order partial differential equation*

$$\begin{cases} -v_t(t,x) + \sup\limits_{u\in U} G(t,x,u,-\nabla_x v(t,x), -\nabla_{xx}v(t,x)) = 0, \quad (t,x)\in[0,T]\times\mathbb{R}^d, \\ v(T,x) = k(x), \quad x\in\mathbb{R}^d, \end{cases} \tag{3.19}$$

where

$$\begin{aligned} G(t,x,u,p,P) \;\; &\equiv \;\; \frac{1}{2}tr(P\sigma\sigma^*(t,x,u)) + \langle p, b(t,x,u)\rangle - f(t,x,u), \\ &\quad\;\; (t,x,u,p,P)\in[0,T]\times\mathbb{R}^d\times U\times\mathbb{R}^d\times\mathbb{S}^d. \end{aligned} \tag{3.20}$$

Proof. See Yong and Zhou [1999], Chapter 4, Proposition 3.5.

□

Equation (3.19) is called HJB equation of Problem (P) and the function G defined in (3.20) is called the *generalized Hamiltonian.*
Note at this point the connection between the HJB equation and the Feynman-Kac formula we reviewed in Chapter 2 in the context of Markovian BSDEs. Without the "sup" and without the control process entering the coefficients, equation (3.19) is identical to equation (2.25). We saw in Proposition 2.5.10 that the solution to PDE (2.25) is equivalent to the solution of an (uncontrolled) Markovian BSDE under the assumption that the solution to equation (2.25) is smooth. Similarly, the preceding Proposition shows that the solution to the PDE (3.19) is equal to the value function of problem (P) under the condition that the solution is smooth. Noting that the value function is the infimum

of the cost functional (3.15) and noting that the cost functional can be written in form of a (controlled) BSDE (see Remark 2.5.9), we see the relation to the Feynman-Kac formula.
In the preceding Proposition, we need to assume that the value function is smooth. Unfortunately, this is not necessarily the case in general. However, under certain assumptions on the coefficients of the stochastic control problem, it is possible to prove the existence of a smooth value function. We refer to Fleming and Rishel [1975], Chapters 6-8 for more information. For the purpose of the work presented here it is sufficient to mention one special situation which is characterized by the following assumption.

Assumption 3.3.4. *Let $d, \ell \in \mathbb{N}$. The noise coefficient σ is assumed not to depend on the control variable u and to be a nonsingular $d \times \ell$-dimensional matrix. Moreover,*

- *U is compact;*
- *$\sigma \in C^{1,2}([0,T] \times \mathbb{R}^d)$ and σ, σ^{-1}, $\nabla_x \sigma$ are bounded on $[0,T] \times \mathbb{R}^d$;*
- *there are functions $\tilde{b} \in C^{1,2}([0,T] \times \mathbb{R}^d)$ and $\theta \in C^{1,2}([0,T] \times \mathbb{R}^d \times U)$ with $\nabla_x \tilde{b}$ bounded on $[0,T] \times \mathbb{R}^d$ and θ, $\nabla_x \theta$ bounded on $[0,T] \times \mathbb{R}^d \times U$, such that*

$$b(s,x,u) = \tilde{b}(s,x) + \sigma(s,x)\theta(s,x,u), \quad (s,x,u) \in [0,T] \times \mathbb{R}^d \times U;$$

- *f is in $C^{1,2}([0,T] \times \mathbb{R}^d \times U)$ and $f, \nabla_x f$ satisfy a polynomial growth condition;*
- *k is in $C^2(\mathbb{R}^d)$ and k, $\nabla_x k$ satisfy a polynomial growth condition.*

The proof of the following theorem can be found in Fleming and Rishel [1975], Chapter 6, Theorem 6.2.

Theorem 3.3.5. *Let Assumption 3.3.4 hold. Then for any $t \in [0,T]$, equation (3.19) has a unique solution $V \in C_p^{1,2}([t,T] \times \mathbb{R}^d)$ with V continuous in $([t,T] \times \mathbb{R}^d)$. Here $C_p^{1,2}([0,T] \times \mathbb{R}^d)$ denotes the set of all functions in $C^{1,2}([0,T] \times \mathbb{R}^d)$ which satisfy a polynomial growth condition on $([0,T] \times \mathbb{R}^d)$.*

The following result shows that under the conditions of Theorem 3.3.5 there exists an optimal Markov strategy (see Definition 3.2.4).

Theorem 3.3.6. *Under the assumptions of Theorem 3.3.5, there exists for all $t \in [0,T]$ an optimal admissible Markov strategy $u^* : [t,T] \times \mathbb{R}^d \to U$, satisfying almost everywhere in $[0,T] \times \mathbb{R}^d$*

$$G(t,x,u^*(t,x), -\nabla_x V(t,x), -\nabla_{xx} V(t,x)) = \sup_{u \in U} G(t,x,u, -\nabla_x V(t,x), -\nabla_{xx} V(t,x)).$$

Proof. For a proof see Fleming and Rishel [1975], Chapter 6, Theorem 6.3.

□

Since the above assumption on the coefficients in Theorem 3.3.5 is quite restrictive, we would like to drop smoothness assumptions while preserving the HJB equation. For this purpose, we will introduce the notion of viscosity solutions, as in the context of the Feynman-Kac formula. Then we will characterize the value function as the unique viscosity solution of the corresponding HJB equation (3.19).

Definition 3.3.7. Suppose $v \in C([0,T] \times \mathbb{R}^d)$ satisfies

$$v(T,x) \leq k(x), \quad x \in \mathbb{R}^d,$$

(resp.

$$v(T,x) \geq k(x), \quad x \in \mathbb{R}^d).$$

Then v is called a *viscosity subsolution* (resp. *supersolution*) of PDE (3.19) if, for any $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d)$, whenever $\varphi - v$ attains a local minimum (resp. local maximum) at $(t,x) \in [0,T) \times \mathbb{R}^d$, we have,

$$-\varphi_t(t,x) + \sup_{u \in U} G(t,x,u,-\nabla_x \varphi(t,x), -\nabla_{xx}\varphi(t,x)) \leq 0$$

(resp.

$$-\varphi_t(t,x) + \sup_{u \in U} G(t,x,u,-\nabla_x \varphi(t,x), -\nabla_{xx}\varphi(t,x)) \geq 0).$$

A function $v \in C([0,T] \times \mathbb{R}^d)$ is called a *viscosity solution* of PDE (3.19) if it is both a viscosity subsolution and a viscosity supersolution of PDE (3.19).

Now we are able to generalize Proposition 3.3.3.

Theorem 3.3.8. *Suppose that Assumptions 3.1.1 and 3.2.1 hold. Then the value function V is a viscosity solution of the HJB equation (3.19).*

Proof. For a proof, we refer to Yong and Zhou [1999], Chapter 4, Theorem 5.2.

□

The notion of solution in the viscosity sense is much weaker than the classical one. Thus it is easier to find a solution in the viscosity sense. The issue of existence is usually not a problem. In fact, with the preceding Theorem, we have proved existence of a viscosity solution of the HJB equation (3.19) by showing that the value function is one. Besides the existence of solutions, it is important to have information on uniqueness in a class of functions, so that the HJB equation effectively characterizes the value function. Fortunately, uniqueness does hold. For a proof of the following theorem see Yong and Zhou [1999], Chapter 4, Theorem 6.1.

Theorem 3.3.9. *Let Assumptions 3.1.1 and 3.2.1 hold. Then the HJB equation (3.19) admits at most one viscosity solution v in the class of functions satisfying*

$$|v(t,x)| \leq K(1 + |x|), \quad (t,x) \in [0,T] \times \mathbb{R}^d,$$

and

$$|v(t,x)-v(t',x')| \leq K\left(|x-x'| + (1+|x|+|x'|)|t-t'|^{1/2}\right), \quad (t,x),(t',x') \in [0,T]\times\mathbb{R}^d.$$

Consequently, we obtain uniqueness of a viscosity solution that is Lipschitz continuous in the state variable and (1/2)-Hölder continuous in the time variable.
Rather than starting with the optimal control, and showing that the HJB equation follows, it is also possible to start with the HJB equation (regarded simply as a nonlinear PDE) and suppose that a solution exists. It can be shown that this solution does indeed coincide with the value function of an optimal control problem, and that the control strategy obtained from the maximum in the HJB equation is indeed optimal. This procedure is called *verification*, and is extremely practical. It says that if we can actually find a smooth solution to the HJB equation, this solution gives an optimal control, which is what we care about in practice.
However, within this work we will not use verification theory for the application to the optimal liquidation problem. Instead, we will employ another powerful theory that helps to solve stochastic control problems in practice, the so-called *stochastic maximum principle* which will be introduced in the next section. Therefore we refer to Yong and Zhou [1999] to find more details about the verification theory and its applications.

3.4. The Stochastic Maximum Principle

The maximum principle for deterministic control problems was first formulated and derived by Pontryagin in the 1950s. Later in the 1970s Kushner [1972], Haussmann [1976], Bismut [1978] and Bensoussan [1981] extended the maximum principle to *stochastic* control problems. One restriction of these first approaches is that all the results on the stochastic maximum principle were obtained basically under the assumption that the diffusion coefficient is independent of the control. In this case the results did not vary much from the deterministic maximum principle. Later the maximum principle was extended to the case were the control enters the diffusion coefficient of the state process. We mention especially the work of Peng [1990] and Zhou [1991] in this context.
The stochastic maximum principle basically states that any optimal control must solve the so-called Hamiltonian system, which is a system of forward-backward stochastic differential equations, plus a maximum condition of a function called the Hamiltonian. The advantage of the maximum principle over other methods for the solution of stochastic optimal control problem is that it leads to the generally easy task of maximizing the Hamiltonian and that it delivers closed-form solutions in certain classes of optimal control problems.
Within this work, we want to apply the stochastic maximum principle to an optimal execution problem that we will introduce in detail in Chapter 6. The considered optimal execution problem will be formulated as a stochastic control problem for which the diffusion part of the state variable is independent of the control variable. In this case, the stochastic maximum principle is easier to derive and the Hamiltonian system that needs to be solved takes a much easier form than in the general case in which the control

enters the diffusion coefficients. An overview of the more general form of the stochastic maximum principle can be found in Yong and Zhou [1999].

In the following, we will consider a strong formulation of the stochastic optimal control problem. Therefore, we assume that the processes are defined on a given ℓ-dimensional Brownian basis $\nu = (\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W)$. Note that in this section we only consider strict controls. The maximum principle for relaxed control problems is established in Bahlali and Mezerdi [2002] and Bahlali et al. [2006].
As before we assume that some finite time horizon T is given, we assume that $t \in [0,T]$ and that U is a separable metric space. We will impose the following assumption on the coefficients of our control problem.

Assumption 3.4.1. *Let $d, \ell \in \mathbb{N}$. The maps $b : [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}^d$, $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times\ell}$, $f : [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}$ and $k : \mathbb{R}^d \to \mathbb{R}$ are measurable and twice continuously differentiable with respect to x. They and all their derivatives $\nabla_x b$, $\nabla_{xx} b$, $\nabla_x \sigma$, $\nabla_{xx}\sigma$, $\nabla_x f$, $\nabla_{xx} f$, $\nabla_x k$ and $\nabla_{xx} k$ are continuous in (x,u) for all $(x,u) \in \mathbb{R}^d \times U$. $\nabla_x b$, $\nabla_{xx} b$, $\nabla_x \sigma$, $\nabla_{xx}\sigma$, $\nabla_{xx} f$ and $\nabla_{xx} k$ are bounded and there is a constant $K > 0$ such that for all $s \in [0,T]$, $u \in U$ and $x \in \mathbb{R}^d$, $\nabla_x f(s,x,u)$ is bounded by $K(1+|x|+|u|)$ and $\nabla_x k(x)$ is bounded by $K(1+|x|)$.*

Let $(t,x) \in [0,T] \times \mathbb{R}^d$ and let $u \in \mathcal{U}_\nu[t,T]$. Then, the coefficients enter the state process

$$dX_s^{t,x,u} = b(s, X_s^{t,x,u}, u_s)ds + \sigma(s, X_s^{t,x,u})dW_s, \quad X_t^{t,x,u} = x, \quad s \in [t,T], \tag{3.21}$$

as well as the cost functional

$$J(t,x,u) = E\left[\int_t^T f(s, X_s^{t,x,u}, u_s)ds + k(X_T^{t,x,u})\right]. \tag{3.22}$$

From Theorem 3.1.2 we see that under Assumption 3.4.1, for any $u \in \mathcal{U}_\nu[t,T]$, the state equation (3.21) admits a unique solution $(X^{t,x,u})_{(t,x)\in[0,T]\times\mathbb{R}^d}$.
Let us for clarity redefine the aim of the stochastic control problem.

Problem (P): Minimize (3.22) subject to (3.21) over $\mathcal{U}_\nu[t,T]$, $t \in [0,T]$.

As before (compare Definition 3.2.3), we denote the optimal control as $\bar{u}$ and the corresponding state process as $(\bar{X}^{t,x})_{(t,x)\in[0,T]\times\mathbb{R}^d}$. In order to simplify the notation, we will omit for the rest of this section the superscripts in $(\bar{X}^{t,x})_{(t,x)\in[0,T]\times\mathbb{R}^d}$ if there is no need to underline the dependence of the solution on t and x. Therefore, we will write for $(t,x) \in [0,T] \times \mathbb{R}^d$

$$\bar{X}_s \equiv \bar{X}_s^{t,x}, \quad s \in [t,T].$$

The main "ingredients" of the stochastic maximum principle besides the state process (3.21) and the performance functional (3.22) are the so called *adjoint equation* and the *Hamiltonian*, which we will define now.

The adjoint equation takes the form of a Markovian backward stochastic differential equation. For all $t \in [0,T]$,

$$\begin{cases} dp_s = -\Big(\nabla_x b(s,\bar{X}_s,\bar{u}_s)^* p_s + \sum_{j=1}^{\ell} \nabla_x \sigma^j(s,\bar{X}_s)^* q_s^j \\ \qquad\qquad -\nabla_x f(s,\bar{X}_s,\bar{u}_s)\Big)\, ds + q_s dW_s, \quad s \in [t,T], \\ p_T = -\nabla_x k(\bar{X}_T), \end{cases} \tag{3.23}$$

were σ^j, resp. q^j, $1 \le j \le \ell$, denotes the j-th column of matrix σ, resp. q. Because of Assumption 3.4.1, we know from Proposition 2.5.7 that equation (3.23) admits a unique adapted solution $(p,q) \in \mathcal{S}^2_\nu([t,T];\mathbb{R}^d) \times \mathcal{H}^2_\nu([t,T];\mathbb{R}^{d\times\ell})$.

The solution of the state process and the adjoint equation enter the so called *Hamiltonian*, which is defined by

$$H(s,x,u,p,q) = tr\left(q^*\sigma(s,x)\right) + \langle p, b(s,x,u)\rangle - f(s,x,u), \qquad (s,x,u,p,q) \in [0,T]\times\mathbb{R}^d\times U\times\mathbb{R}^d\times\mathbb{R}^{d\times\ell}. \tag{3.24}$$

Note here the small difference between the just defined Hamiltonian and the *generalized* Hamiltonian G which was defined in (3.20) in the context of the HJB equation in the preceding chapter.
With these prerequisites, we are now able to formulate the stochastic maximum principle.

Theorem 3.4.2. *Let Assumption 3.4.1 hold. Let $(\bar{X},\bar{u})$ be an optimal pair for Problem (P). Then, for any $t \in [0,T]$, there is a pair of processes $(p,q) \in \mathcal{S}^2_\nu([t,T];\mathbb{R}^d) \times \mathcal{H}^2_\nu([t,T];\mathbb{R}^{d\times\ell})$, satisfying the adjoint equations (3.23), such that*

$$H(s,\bar{X}_s,\bar{u}_s,p_s,q_s) = \max_{u\in U} H(s,\bar{X}_s,u_s,p_s,q_s), \text{ for a.e. } s\in[t,T], \text{ } P\text{-a.s.} \tag{3.25}$$

Proof. For a proof see Peng [1990], Theorem 3. There the more general case is considered where the diffusion coefficient enters the control variable. It is easy to check that the statement of the theorem takes the above form if the diffusion does not contain the control variable.

□

Note that for any $t \in [0,T]$ the state process (3.21) and the adjoint process (3.23) can be written as a fully coupled controlled FBSDE taking the following form for any $s \in [t,T]$:

$$\begin{cases} dX_s = H_p(s,X_s,u_s,p_s,q_s)ds + H_q(s,X_s,u_s,p_s,q_s)dW_s, \quad X_t = x, \\ \\ dp_s = -H_x(s,X_s,u_s,p_s,q_s)ds + q_s dW_s, \quad p_T = -k_x(X_T). \end{cases} \tag{3.26}$$

Equations (3.25) and (3.26) together are called *stochastic Hamiltonian system*. Its solution is a 4-tuple (X,u,p,q). We denote the optimal 4-tuple as $\left(\bar{X},\bar{u},p,q\right)$.

Theorem 3.4.2 provides necessary conditions to be fulfilled by the optimal solution. In order to apply the stochastic maximum principle, we are interested in sufficient conditions of optimality. To obtain these, we need some additional convexity/concavity conditions.

Theorem 3.4.3. *(Sufficient Conditions of Optimality) Let $t \in [0,T]$ and let Assumption 3.4.1 hold. In addition we assume that U is a convex body (i.e., U is convex and has nonempty interior). Let $\left(\bar{X}, \bar{u}, p, q\right)$ be an admissible 4-tuple. Suppose that k is convex and $H(s,\cdot,\cdot,p_s,q_s)$ is concave for all $s \in [t,T]$ almost surely, and*

$$H(s,\bar{X}_s,\bar{u}_s,p_s,q_s) = \max_{u \in U} H(s,\bar{X}_s,u,p_s,q_s), \textit{for a.e. } s \in [t,T], \; P\textit{-a.s.} \tag{3.27}$$

Then $(\bar{X},\bar{u})$ is an optimal pair of Problem (P).

Proof. For $(t,x) \in [0,T] \times \mathbb{R}^d$ we want to show, that for any admissible control $u \in U_\nu[t,T]$, we have

$$J(t,x,\bar{u}) \leq J(t,x,u).$$

This is equivalent to

$$E\left[\int_t^T f(s,\bar{X}_s,\bar{u}_s)ds + k(\bar{X}_T)\right] \leq E\left[\int_t^T f(s,X_s,u_s)ds + k(X_T)\right], \quad t \in [0,T],$$

and this again is equivalent to the inequality

$$E\left[\int_t^T (f(s,\bar{X}_s,\bar{u}_s) - f(s,X_s,u_s))ds\right] + E\left[k(\bar{X}_T) - k(X_T)\right] \leq 0, \quad t \in [0,T]. \tag{3.28}$$

We will consider in the following the two expressions above separately. For that we define

$$D_1 \equiv E\left[\int_t^T (f(s,\bar{X}_s,\bar{u}_s) - f(s,X_s,u_s))ds\right], \quad t \in [0,T],$$

and

$$D_2 \equiv E\left[k(\bar{X}_T) - k(X_T)\right].$$

Using the definition of the Hamiltonian (compare Definition 3.24), we can describe D_1 for $t \in [0,T]$ as

$$\begin{aligned} D_1 &= E\left[\int_t^T -(H(s,\bar{X}_s,\bar{u}_s,p_s,q_s) - H(s,X_s,u_s,p_s,q_s))ds\right] \\ &+ E\left[\int_t^T \left\langle b(s,\bar{X}_s,\bar{u}_s) - b(s,X_s,u_s), p_s\right\rangle ds\right] \\ &+ E\left[\int_t^T tr(q_s^*(\sigma(s,\bar{X}_s) - \sigma(s,X_s)))ds\right] \end{aligned}$$

$$=: \quad \Delta_1 + \Delta_2 + \Delta_3.$$

Since for all $t \in [0,T]$, $p \in \mathbb{R}^d$ and $q \in \mathbb{R}^{d\times\ell}$, $(x,u) \to H(t,x,u,p,q)$ is concave, we obtain for any $(\bar{x},\bar{u}),(x,u) \in \mathbb{R}^d \times U$,

$$H(t,x,u,p,q) - H(t,\bar{x},\bar{u},p,q) \leq \langle H_x(t,\bar{x},\bar{u},p,q), x-\bar{x}\rangle + \langle H_u(t,\bar{x},\bar{u},p,q), u-\bar{u}\rangle\,, \qquad (t,p,q) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^{d\times\ell}.$$

Consequently, for any $t \in [0,T]$,

$$\begin{aligned}
\Delta_1 &\leq E\left[\int_t^T \left(\left\langle H_x(s,\bar{X}_s,\bar{u}_s,p_s,q_s), X_s-\bar{X}_s\right\rangle + \left\langle H_u(s,\bar{X}_s,\bar{u}_s,p_s,q_s), u_s-\bar{u}_s\right\rangle\right) ds\right] \\
&= E\left[\int_t^T \left\langle H_x(s,\bar{X}_s,\bar{u}_s,p_s,q_s), X_s-\bar{X}_s\right\rangle ds\right].
\end{aligned}$$

Note that in the last step we used (3.27) which implies that $H_u(s,\bar{X}_s,\bar{u}_s,p_s,q_s) = 0$. Using (3.26) yields for any $t \in [0,T]$

$$\begin{aligned}
\Delta_1 &\leq E\left[\int_t^T -\left\langle X_s-\bar{X}_s, dp_s\right\rangle + \int_t^T \left\langle X_s-\bar{X}_s, q_s dW_s\right\rangle\right] \\
&= E\left[\int_t^T -\left\langle X_s-\bar{X}_s, dp_s\right\rangle\right]. \qquad (3.29)
\end{aligned}$$

Next we consider D_2. Since k is convex, we have

$$\begin{aligned}
D_2 &= E\left[k(\bar{X}_T) - k(X_T)\right] \\
&\leq E\left[\left\langle \nabla_x k(\bar{X}_T), \bar{X}_T - X_T\right\rangle\right] \\
&= E\left[\left\langle -p_T, \bar{X}_T - X_T\right\rangle\right] = E\left[\left\langle p_T, X_T-\bar{X}_T\right\rangle\right] \\
&= E\left[\int_t^T \left\{\left\langle X_s-\bar{X}_s, dp_s\right\rangle + \left\langle p_s, d(X_s-\bar{X}_s)\right\rangle\right.\right. \\
&+ \left.\left. tr(q_s^*(\sigma(s,X_s)-\sigma(s,\bar{X}_s)))ds\right\}\right] \\
&= E\left[\int_t^T \left\{\left\langle X_s-\bar{X}_s, dp_s\right\rangle + \left\langle p_s, b(s,X_s,u_s)-b(s,\bar{X}_s,\bar{u}_s)\right\rangle\right.\right. \\
&+ \left.\left. tr(q_s^*(\sigma(s,X_s)-\sigma(s,\bar{X}_s)))ds\right\}\right] \\
&\leq -\Delta_1 - \Delta_2 - \Delta_3.
\end{aligned}$$

Note, that in the last equality we used (3.29). It follows that

$$D_2 \leq -\Delta_1 - \Delta_2 - \Delta_3 = -D_1,$$

and therefore

$$J(t, x, \bar{u}) - J(t, x, u) = D_1 + D_2 \leq 0,$$

which proves (3.28).

□

4. Special Cases of Stochastic Control Problems

In Chapter 1 we introduced the optimal execution problem. We argued that it is reasonable to formulate the problem as a stochastic control problem where the stock price is modelled by a stochastic process in which the control variable, i.e., the trading speed, enters linearly. Due to the transient trading impact the stock price process is assumed to contain some exponential delay effect in consequence of which the present stock price is influenced by past trading decisions. In addition, since the objective is to find the trading strategy that maximizes the expected average selling price, we assume the cost functional of the control problem to be linear in the control variable as well.
Consequently, the optimal execution problem can be formulated as a stochastic control problem with two special characteristics. On the one hand it is a control problem with an exponential delay in the control variable, and on the other hand a control problem with coefficients that are linear in the control variable. In the following two sections, we will consider each of these two special situations separately in more detail.
During this chapter, we will for simplicity consider only strong formulations of control problems with strict controls. Therefore, we assume a certain ℓ-dimensional Brownian stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W)$ to be given. In the following, let $T > 0$ be a finite time horizon and $t \in [0, T]$. As before u denotes a control process which takes values in a separable metric space U.

4.1. Stochastic Control Problems with Exponential Delay

Stochastic control problems that incorporate some delay effect are of great interest in recent literature. The advantage of incorporating time delays is to allow for non-instantaneous interactions which can be observed very often in reality, for example in population models (see Chang [2008]) or advertising models (see Gozzi and Marinelli [2006]). Generally, it is assumed that the delay effect on the present value of the state process does not enter via the control variable, but that instead the evolution of the system is influenced by the states taken in the past some time before. Such processes are called *Stochastic Delay Differential Equations (SDDEs)*. The main difficulty of stochastic control problems with delay is the fact that these problems are in general infinite-dimensional. However, it turns out that for a special structure of the delay component, one can overcome the problem by reducing it to finite dimension. This special situation is given if the state equation depends on the current state, on the state δ time units earlier and on some moving average of the previous states. To be more precise, we need the moving average of the previous states to take an exponential form. Such stochastic

control problems with exponential delay are for example considered in Bauer and Rieder [2005], Elsanosi et al. [2000], Larssen [2002], or Oksendal and Sulem [2001].
We will show in this section that for stochastic control problems with an exponential delay in the control variable, it is possible to do a transformation into a problem with a Markovian state process.

For $(t,x) \in [0,T] \times \mathbb{R}^d$ and any admissible control $u \in \mathcal{U}_\nu[t,T]$ consider the controlled stochastic differential equation

$$\begin{cases} dX_s = b(s,X_s,Y_s,u_s)ds + \sigma(s,X_s,Y_s,u_s)dW_s, \quad s \in [t,T], \\ X_t = x, \end{cases} \tag{4.1}$$

with

$$Y_s \equiv \int_{-\infty}^0 e^{\lambda\tau} g(u_{s+\tau})d\tau.$$

Here, $g : U \to \mathbb{R}^m$, $m \in \mathbb{N}$, is continuously differentiable and $\lambda \in \mathbb{R}$ is a positive constant. Equation (4.1) contains a measurable drift function

$$b : [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \times U \to \mathbb{R}^d,$$

and a measurable volatility function

$$\sigma : [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \times U \to \mathbb{R}^{d\times\ell}.$$

Note that for simplicity we omit the superscripts in (4.1) that underline the dependence of the solution on t, x and u.
For any $t \in [0,T]$ we define

$$u_s = 0 \text{ for all } s \le t.$$

This means that the controller starts having an influence on the state variable at starting time $t \in [0,T]$ and not before.
The performance functional of the stochastic control problem is given by

$$J(t,x,u) = E\left[\int_t^T f(s,X_s,Y_s,u_s)ds + k(X_T,Y_T)\right], \quad (t,x) \in [0,T] \times \mathbb{R}^d, u \in \mathcal{U}_\nu[t,T], \tag{4.2}$$

with $f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \times U \to \mathbb{R}$ and $k : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$.
Before we give necessary conditions on the coefficients that ensure the existence of a solution of (4.1) and ensure that the cost functional is well defined, we will show how in this special case the state process can be transformed into an ordinary (Markovian) stochastic differential equation.
For $t \in [0,T]$, $u \in \mathcal{U}_\nu[t,T]$ and $s \in [t,T]$, define $G_s \equiv \int_t^s g(u_\tau)d\tau$. Then by integration by parts

$$Y_s = \int_{-\infty}^0 e^{\lambda\tau} g(u_{s+\tau})d\tau \quad = \quad e^{\lambda\tau} G_{s+\tau}|_{-\infty}^0 - \int_{-\infty}^0 \lambda e^{\lambda\tau} G_{s+\tau} d\tau$$

$$= \quad G_s - \int_{-\infty}^{0} \lambda e^{\lambda\tau} G_{s+\tau} d\tau.$$

Since Y is of finite variation, we get

$$dY_s = (g(u_s) - \lambda Y_s)\, ds. \tag{4.3}$$

Now, by combining differential equation (4.3) with the system of SDEs (4.1), we obtain a $(d+m)$-dimensional process

$$\tilde{X} \equiv \begin{pmatrix} X \\ Y \end{pmatrix},$$

which is described by the following SDE for $(t, x) \in [0, T] \times \mathbb{R}^d$ and $u \in \mathcal{U}_\nu[t, T]$:

$$\begin{cases} d\tilde{X}_s = \tilde{b}(s, \tilde{X}_s, u_s)ds + \tilde{\sigma}(s, \tilde{X}_s, u_s)dW_s, & s \in [t, T], \\ \tilde{X}_t = (X_t, Y_t) \equiv \tilde{x}, \end{cases} \tag{4.4}$$

with $\tilde{b} : [0, T] \times \mathbb{R}^{d+m} \times U \to \mathbb{R}^{d+m}$ and $\tilde{\sigma} : [0, T] \times \mathbb{R}^{d+m} \times U \to \mathbb{R}^{(d+m)\times\ell}$ given by

$$\tilde{b}(s, \tilde{x}, u) \equiv \begin{pmatrix} b(s, \tilde{x}, u) \\ g(u) - \lambda y \end{pmatrix} \quad \text{and} \quad \tilde{\sigma}(s, \tilde{x}, u) \equiv \begin{pmatrix} \sigma(s, \tilde{x}, u) \\ \mathbf{0} \end{pmatrix},$$

where $\tilde{x} = (x, y)$ with $x \in \mathbb{R}^d$, $y \in \mathbb{R}^m$ and $\mathbf{0}$ denotes an $m \times \ell$-dimensional matrix consisting of zeros.
Using the transferred process $\tilde{X}$, the cost functional (4.2) takes the form

$$J(t, \tilde{x}, u) = E\left[\int_t^T f(s, \tilde{X}_s, u_s)ds + k(\tilde{X}_T)\right], \quad (t, \tilde{x}) \in [0, T] \times \mathbb{R}^{d+m}, u \in \mathcal{U}_\nu[t, T]. \tag{4.5}$$

Consequently, we have shown that given the special situation of an exponential delay in the control variable, we are able to transform the SDDE (4.1) into a Markovian SDE (4.4) with higher dimensional state space. With the transformation the delay process also vanishes from the performance functional, such that we can apply the stochastic maximum principle for Markovian processes as summarized in the preceding chapter.

Remark 4.1.1. Note that we pay some price for removing the explicit delay. We obtain a degenerate diffusion process.

It is easy to see that if the coefficients b and σ of the SDE (4.1) fulfill the usual Lipschitz and growth condition (compare Assumption 3.1.1), then the same holds for the functions $\tilde{b}$ and $\tilde{\sigma}$, such that a unique strong solution to (4.1) exists for any admissible control process u. If in addition the coefficients f and k of the cost functional (4.2) fulfill the Lipschitz and growth condition as well (compare Assumption 3.2.1), the cost functional is well defined.

4.2. Stochastic Bang-Bang Problems

In this section we will discuss stochastic control problems with coefficients that are all linear with respect to the control variable. We will show that problems of this kind lead to an optimal control process that has "bang-bang" character. This means that the optimal control process is not continuous, but switches abruptly between extreme states. This discontinuity of the control process leads to a variety of problems as we will see later in this section. Let us start with presenting the appropriate setting for stochastic control problems with bang-bang solutions.
We will impose the following assumption on the coefficients of the control problem considered now.

Assumption 4.2.1. *The functions $b^1, b^2 : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times\ell}$, $f^1, f^2 : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, and $k : \mathbb{R}^d \to \mathbb{R}$ are measurable and continuously differentiable with respect to x. The derivatives $\nabla_x b^1$, $\nabla_x b^2$ and $\nabla_x \sigma$ are uniformly bounded and there is a constant $K > 0$ such that for all $(s,x) \in [0,T] \times \mathbb{R}^d$, $\nabla_x f^1(s,x)$, $\nabla_x f^2(s,x)$ and $\nabla_x k(x)$ are bounded by $K(1+|x|)$.*

Remark 4.2.2. Note that Assumption 4.2.1 implies that the functions $b^1, b^2, \sigma, f^1, f^2$ and k fulfill the usual Lipschitz and linear growth condition, i.e., there exists a constant $K > 0$ such that for all $s \in [0,T]$ and $x, y \in \mathbb{R}^d$ we have

$$\begin{cases} |b^1(s,x) - b^1(s,y)| + |b^2(s,x) - b^2(s,y)| + \|\sigma(s,x) - \sigma(s,y)\| \\ +|f^1(s,x) - f^1(s,y)| + |f^2(s,x) - f^2(s,y)| + |k(x) - k(y)| \le K|x-y|, \\ \\ |b^1(s,x)| + |b^2(s,x)| + \|\sigma(s,x)\| + |f^1(s,x)| + |f^2(s,x)| + |k(x)| \le K(1+|x|). \end{cases}$$

Assume now that the considered state process looks as follows for $(t,x) \in [0,T] \times \mathbb{R}^d$ and $u \in \mathcal{U}_\nu[t,T]$:

$$\begin{cases} dX_s = (b^1(s,X_s) + b^2(s,X_s)u_s)\, ds + \sigma(s,X_s) dW_s, \quad s \in [t,T], \\ X_t = x. \end{cases} \tag{4.6}$$

Under Assumption 4.2.1 the above SDE has a unique solution $X^{t,x,u}$. The process $X^{t,x,u}$ enters the cost functional of the form

$$J(t,x,u) = E\left[\int_t^T \left(f^1(s, X_s^{t,x,u}) + f^2(s, X_s^{t,x,u}) u_s\right) ds + k(X_T^{t,x,u})\right], \tag{4.7}$$
$$(t,x) \in [0,T] \times \mathbb{R}^d, u \in \mathcal{U}_\nu[t,T].$$

We will impose the following additional assumption on the set U.

Assumption 4.2.3. *For any $t \in [0,T]$, the control process takes values in the compact set*

$$U = [C_1, C_2],$$

where $C_1, C_2 \in \mathbb{R}$ are finite constants.

Remark 4.2.4. We will see that the optimal solution of this specific control problem (provided that it exists) takes the form of a discontinuous process that jumps between the just defined extreme states C_1 and C_2. Therefore, the optimal solution obviously depends heavily on the choice of these finite constants.

Since all coefficients of the considered control problem are linear in the control variable u, we obtain a corresponding Hamiltonian that is linear in u as well, namely

$$\begin{aligned} & H(s,x,u,p,q) \\ = \; & -f^1(s,x) - f^2(s,x)u + \left\langle p, b^1(s,x) + ub^2(s,x) \right\rangle + tr\left[q^*\sigma(s,x)\right] \\ = \; & u\left(-f^2(s,x) + \left\langle p, b^2(s,x)\right\rangle\right) - f^1(s,x) + \left\langle p, b^1(s,x)\right\rangle + tr\left[q^*\sigma(s,x)\right], \\ & (s,x,u,p,q) \in [0,T] \times \mathbb{R}^d \times U \times \mathbb{R}^d \times \mathbb{R}^{d\times \ell}. \end{aligned}$$

Let us now consider the question of existence of an optimal solution to the described problem. Let us assume that an optimal solution $\bar{u}$ and the corresponding solution $X^{t,x,\bar{u}}$ of equation (4.6) exist. As before, we denote the optimal pair as $(\bar{X}^{t,x}, \bar{u}) := (X^{t,x,\bar{u}}, \bar{u})$ and if there is no need to underline the dependence of the solution on t and x, we will write $\bar{X}^{t,x} = \bar{X}$. From the stochastic maximum principle we know that the optimal control (if it exists) will maximize the Hamiltonian. Due to the linearity of the Hamiltonian with respect to u, it follows that the optimal value for u switches discontinuously between the two extreme states C_1 and C_2 depending on the sign of the coefficient of u in the Hamiltonian. This means that the optimal control would be of the form

$$\bar{u}_s = \begin{cases} C_1, \text{ if } -f^2(s,\bar{X}_s) + \left\langle p_s, b^2(s,\bar{X}_s)\right\rangle \leq 0, \\ C_2, \text{ if } -f^2(s,\bar{X}_s) + \left\langle p_s, b^2(s,\bar{X}_s)\right\rangle > 0. \end{cases} \tag{4.8}$$

Such a process which jumps only between two values is called *bang-bang process*. Therefore, a stochastic control problem with coefficients that are all linear with respect to the control variable, is called in general a *stochastic bang-bang problem*. From (4.8) we can see that, given $t \in [0,T]$, it is possible to define a measurable function $v : [t,T] \times \mathbb{R}^d \times \mathbb{R}^d \to U$ such that $\bar{u}_s = v(s, \bar{X}_s, p_s)$ for every $s \in [t,T]$. It is clear that the function v is discontinuous in general.
For $t \in [0,T]$ consider now the corresponding Hamiltonian system consisting of the state process and the adjoint variable (compare equation (3.26))

$$\begin{cases} dX_s = \left(b^1(s,X_s) + b^2(s,X_s)u_s\right) ds + \sigma(s,X_s)dW_s, \\ dp_s = \big(\nabla_x f^1(s,X_s) + \nabla_x f^2(s,X_s)u_s - \langle p, \nabla_x b^1(s,X_s) + u_s \nabla_x b^2(s,X_s)\rangle \\ \qquad\qquad -tr\left[q^T \nabla_x \sigma(s,X_s)\right]\big) ds + q_s dW_s, \quad s \in [t,T], \\ X_t = x, \quad p_T = -\nabla_x k(X_T). \end{cases}$$

Note that this system is a controlled decoupled FBSDE. By plugging the optimal control in form of the function v into the above system, we obtain for any $(t,x) \in [0,T] \times \mathbb{R}^d$

the uncontrolled fully coupled FBSDE with discontinuous coefficients

$$\begin{cases} d\bar{X}_s = \Big(b^1(s,\bar{X}_s) + b^2(s,\bar{X}_s)v(s,\bar{X}_s,p_s)\Big)\, ds + \sigma(s,\bar{X}_s)dW_s, \quad s \in [t,T], \\ dp_s = \Big(\nabla_x f^1(s,\bar{X}_s) + \nabla_x f^2(s,\bar{X}_s)v(s,\bar{X}_s,p_s) \\ \qquad - \Big\langle p, \nabla_x b^1(s,\bar{X}_s) + v(s,\bar{X}_s,p_s)\nabla_x b^2(s,\bar{X}_s)\Big\rangle - tr\left[q^T \nabla_x \sigma(s,\bar{X}_s)\right]\Big)\, ds + q_s dW_s, \\ \bar{X}_t = x, \quad p_T = -\nabla_x k(\bar{X}_T). \end{cases}$$

We now see that the question of existence of an optimal solution to the stochastic bang-bang problem is equivalent to the question of the solvability of the above FBSDE. If a solution $(\bar{X}, p, q)$ to the above FBSDE exists, the optimal control is determined by plugging the solution into the function v, i.e., $\bar{u}_s = v(s,\bar{X}_s,p_s)$ for every $s \in [t,T]$. Obviously, the discontinuity of the coefficients makes it very difficult to handle the above FBSDE. As mentioned in Remark 2.5.17 the continuity of the coefficients (besides other properties) is essential for proving the existence of a (strong) solution. Also the numerical solution of such problems is very difficult.
For *deterministic* control problems with bang-bang controls, in several attempts numerical problems were tried to overcome by continuation methods. This means that the coefficient of the cost functional is perturbed by some non-linear penalty term. See for example Martinon and Gergaud [2006] or Bertrand and Epenoy [2002] who use a quadratic perturbation, such that the cost functional (4.7) takes the form

$$J^\delta(t,x,u) = E\left[\int_t^T \Big(f^1(s,X_s) + f^2(s,X_s)u_s + \delta u_s^2\Big)\, ds + k(X_T)\right],$$
$$(t,x) \in [0,T] \times \mathbb{R}^d, u \in \mathcal{U}_\nu[t,T],$$

with $\delta \in (0,1]$.
Obviously, by adding a non-linear term to the cost functional, the optimal solution that maximizes the Hamiltonian loses its bang-bang character and becomes smooth. Therefore, one can use standard methods to solve related control problems numerically. It is clear that the influence of the quadratic perturbation depends on the value of δ. For δ converging to 0, the perturbed cost functional converges to the original linear functional and one can show that the continuation procedure yields a good approximation of the initial solution. Bertrand and Epenoy [2002] alternatively also use logarithmic penalty functions, which - depending on the special situation - in cases work better than the quadratic one.
For stochastic bang-bang controls, it seems that there is no literature so far which treats continuation approaches in order to prove the existence of an optimal solution and to find a smooth approximation of the discontinuous optimal control. The following chapter will discuss such an approach.

Remark 4.2.5. It is well known that if the bounds of the control variable are infinite, i.e., $C_1 = -\infty$ and $C_2 = \infty$, then the just described stochastic bang-bang Problem is equivalent to a singular control problem, belonging to a very challenging class of problems in stochastic control theory. To see the connection between a bang-bang problem and a singular control problem, we define the following singular control process.

$$dR_s \equiv u_s ds, \quad s \in [0, T].$$

Roughly speaking, by singular control we mean that the control terms in the dynamics of the state process need not be absolutely continuous with respect to Lebesgue measure and are only required to have paths of bounded variation.
Using the singular control, the state process (4.6) and the cost functional (4.7) take for $(t, x) \in [0, T] \times \mathbb{R}^d$ the form

$$dX_s = b^1(s, X_s)ds + b^2(s, X_s)dR_s + \sigma(s, X_s)dW_s, \quad X_t = x \in \mathbb{R}^d, \quad s \in [t, T],$$

and

$$J(t, x, R) \equiv E\left[\int_t^T f^1(s, X_s)ds + \int_t^T f^2(s, X_s)dR_s + k(X_T)\right].$$

The HJB equations for such problems are typically quite hard to work with. Despite the fact that during the last years there has been a significant development in the theory of weak and viscosity solutions of HJB equations for such diffusion control problems, the existence, uniqueness and regularity theory for this class of PDEs is not well understood. A successful alternative approach is the stochastic maximum principle which can be extended to singular control problems. Cadenillas and Haussmann [1994] presented first results on this area and many other followed. Even for relaxed controls there are results for singular optimal control of diffusions (see for example Bahlali et al. [2007]).
In view of applications to mathematical finance it is particularly important to develop methods for numerical approximations for such control problems. A key method was developed by Kushner and Dupuis [2001], the so called *Markov chain approximation*. The Markov chain approximation method does not require the smoothness of the cost or value function, nor does it rely on uniqueness properties of the associated HJB equations. This is a significant advantage in problems where the PDE theory for the associated HJB equations is hard to tackle.
The interested reader may consult Boetius [2001] for a survey on stochastic singular control problems.

5. Smoothing Stochastic Bang-Bang Problems with Possibly Degenerate Diffusion

Motivated by the optimal execution problem in Chapter 1, we presented in the previous chapter two special cases of stochastic optimal control problems. This was on the one hand a control problem with an exponential delay in the control variable and on the other hand a control problem where the state process as well as the cost functional are linear in the control variable such that the optimal solution has bang-bang character. From the preceding chapter, we know that this will lead to a stochastic control problem facing two challenges. The considered state process will have a degenerate diffusion (see Remark 4.1.1), and due to the discontinuity of the solution the proof of existence of an optimal solution as well as a numerical approach of such a problem will be difficult.
We have two objectives in this chapter. At first, we want to show the existence of an optimal solution, and secondly, we aim at obtaining a smooth approximation for the discontinuous bang-bang solution. We will see that we can reach both goals at once: we construct a sequence of approximating control problems which have continuous optimal solutions that converge to the optimal solution of the original problem.

5.1. The Model

The idea of constructing an approximating sequence of stochastic control problems in order to show the existence of some given stochastic control problem goes back to Buckdahn et al. [2010]. Therefore, we will use a similar notation as they do in this thesis.
Let $T > 0$ be a finite time horizon and let $U = [C_1, C_2]$ with $C_1, C_2 \in \mathbb{R}$ and $C_1 \leq C_2$. Let us fix some initial reference d-dimensional Brownian stochastic basis $\nu = (\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W)$. As before we denote the set of admissible controls with $\mathcal{U}_\nu[t,T]$ (compare Definition 3.2.2).
For any initial condition $(t,x) \in [0,T] \times \mathbb{R}^d$ and any admissible control $u = (u_s)_{s\in[t,T]} \in \mathcal{U}_\nu[t,T]$, we consider the following family of Markovian BSDEs.

$$\begin{cases} dX_s^{t,x,u} = b(X_s^{t,x,u}, u_s)ds + \sigma(X_s^{t,x,u})dW_s, \\ dY_s^{t,x,u} = -f(X_s^{t,x,u}, u_s)ds + Z_s^{t,x,u}dW_s, \quad s \in [t,T], \\ X_t^{t,x,u} = x, \quad Y_T^{t,x,u} = k(X_T^{t,x,u}), \end{cases} \tag{5.1}$$

where the functions

$$b : \mathbb{R}^d \times U \to \mathbb{R}^d, \sigma : \mathbb{R}^d \to \mathbb{S}^d, f : \mathbb{R}^d \times U \to \mathbb{R} \text{ and } k : \mathbb{R}^d \to \mathbb{R}$$

satisfy the following assumption.

Assumption 5.1.1.

1.
   - *There are functions $b^1, b^2 \in C_b^2(\mathbb{R}^d)$ with b^2 being bounded such that for all $(x,u) \in \mathbb{R}^d \times U$, $b(x,u) = b^1(x) + ub^2(x)$;*
   - *$\sigma \in C_b^2(\mathbb{R}^d)$ is bounded and nonnegative definite;*
2.
   - *$k \in C_b^2(\mathbb{R}^d)$;*
   - *there are functions $f^1, f^2 \in C_b^2(\mathbb{R}^d)$ such that for all $(x,u) \in \mathbb{R}^d \times U$, $f(x,u) = f^1(x) + uf^2(x)$.*

Remark 5.1.2. From Assumption 5.1.1 it follows that

- b, σ, f and k are Lipschitz continuous in x, i.e., there exists some constant $C > 0$ such that for all $x, x' \in \mathbb{R}^d$ and $u \in U$

  $$|b(x,u) - b(x',u)| + \|\sigma(x) - \sigma(x')\| + |f(x,u) - f(x',u)| + |k(x) - k(x')| \le C|x - x'|;$$

- there is a constant Λ such that for all $x \in \mathbb{R}^d$

  $$|b^1(x)| + |f^1(x)| + |f^2(x)| + |k(x)| \le \Lambda(1 + |x|).$$

  Consequently, b, f and k grow at most linearly in x.

Note that the diffusion coefficient σ is not assumed to be invertible.

Remark 5.1.3. From Assumption 5.1.1 it follows that the involved coefficients satisfy the so called *Roxin condition* which can be stated in our context as follows.
For all $x \in \mathbb{R}^d$, the set

$$\{(b(x,v), f(x,v)) \,|v \in U\}$$

is convex.

It follows from Proposition 2.5.7 that under Assumption 5.1.1 the system (5.1) has a unique solution $(X^{t,x,u}, Y^{t,x,u}, Z^{t,x,u}) \in \mathcal{S}_\nu^2([t,T];\mathbb{R}^d) \times \mathcal{S}_\nu^2([t,T];\mathbb{R}) \times \mathcal{H}_\nu^2([t,T];\mathbb{R}^d)$ for any $(t,x) \in [0,T] \times \mathbb{R}^d$ and any given admissible control process u.
Note that since the driver does not depend on (y,z) in our case the BSDE (5.1) can be written for any $(t,x) \in [0,T] \times \mathbb{R}^d$ and $u \in \mathcal{U}_\nu[t,T]$ in the form (compare Remark 2.5.9)

$$Y_s^{t,x,u} = E\left[\int_s^T f(X_r^{t,x,u}, u_r)dr + k(X_T^{t,x,u})|\mathcal{F}_s\right], \quad s \in [t,T].$$

We see that the BSDE represents the cost functional of a classical stochastic control problem. Therefore, we define for any $(t,x) \in [0,T] \times \mathbb{R}^d$ and $u \in \mathcal{U}_\nu[t,T]$

$$J(t,x,u) := Y_t^{t,x,u} = E\left[\int_t^T f(X_r^{t,x,u}, u_r)dr + k(X_T^{t,x,u})|\mathcal{F}_t\right]. \tag{5.2}$$

As usual for classical control problems the aim is to minimize this cost functional over all adapted control processes u taking their values in the fixed metric space $U = [C_1, C_2]$ and we introduce the value function V of the control problem as

$$V(t,x) = \inf_{u \in \mathcal{U}_\nu[t,T]} Y_t^{t,x,u}.$$

From Theorem 3.3.8 and Theorem 3.3.9 we know that $V(t,x)$ is continuous in (t,x) and solves in viscosity sense the following Hamilton-Jacobi-Bellman equation:

$$\begin{cases} -V_t(t,x) + \sup\limits_{v \in U} G\left(x, v, -\nabla_x V(t,x), -\nabla_{xx} V(t,x)\right) = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^d, \\ V(T,x) = k(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{5.3}$$

with

$$G(x,v,p,P) = \frac{1}{2} tr\left(P\sigma(x)\sigma(x)^*\right) + \langle p, b(x,v)\rangle - f(x,v),$$

for all $(x,v,p,P) \in \mathbb{R}^d \times U \times \mathbb{R}^d \times \mathbb{S}^d$.
It follows from the stochastic maximum principle (see Theorem 3.4.2) that the optimal control strategy maximizes the Hamiltonian function which under Assumption 5.1.1 takes the form (compare equation (3.24))

$$H(x,v,p,P) = tr\left(P^*\sigma(x)\right) + \left\langle p, b^1(x) + vb^2(x)\right\rangle - f^1(x) - vf^2(x), \tag{5.4}$$
$$(x,v,p,P) \in \mathbb{R}^d \times U \times \mathbb{R}^d \times \mathbb{S}^d.$$

Since the Hamiltonian function is linear in u, the optimal solution has bang-bang character and switches depending on the sign of the coefficient of u in the Hamiltonian discontinuously between the two extreme states C_1 and C_2. Our aim is now to construct a sequence of approximating control systems with coefficients that are convex approximations of the linear coefficients of the original problem. By proving the convergence of the solutions of this sequence to the solution of the original control problem, we obtain two important results. On the one hand we prove the existence of a solution. On the other hand we obtain a smooth approximation of the discontinuous bang-bang solution of the original problem. It happens that we can use this smooth approximation for numerical results in Chapter 6.
The construction of the approximating sequence of control problems will be done similarly as in Buckdahn et al. [2010]. However, there is a couple of differences between our model and the model of Buckdahn et al.. Buckdahn et al. assume that besides the state process the diffusion coefficient σ depends also on the control variable and

consider the more general case, where the driver of the BSDE depends on (y, z). In addition they assume the function b, f and k to be bounded, whereas we only impose a linear growth condition on these functions. In contrast to us, Buckdahn et al. do not necessarily assume the coefficients to be smooth. Consequently, Buckdahn et al. construct an approximating sequence of control problems with smooth approximations of the coefficients.

5.2. An Approximating Control Problem

We define an approximating control problem by substituting the (possibly) degenerate diffusion matrix σ by

$$\sigma^\delta(x) \equiv \sigma(x) + \delta I_d, \tag{5.5}$$

where $\delta \in (0, 1]$ and $x \in \mathbb{R}^d$. Note that the matrix σ^δ is invertible. Obviously, for all $x \in \mathbb{R}^d$ and $\delta, \delta' \in (0, 1]$ we have

$$\left\|\sigma^\delta - \sigma\right\|(x) = \delta, \tag{5.6}$$

and

$$\left\|\sigma^\delta - \sigma^{\delta'}\right\|(x) = |\delta - \delta'|. \tag{5.7}$$

In order to obtain a concave approximation of the linear Hamiltonian, we will introduce the following functions. For each $\delta \in (0, 1]$ and $(x, u) \in \mathbb{R}^d \times U$, we define

$$f^\delta(x, u) = f(x, u) + \delta u^2. \tag{5.8}$$

It follows that for $K = \max(C_1^2, C_2^2)$ the following holds for all $\delta, \delta' \in (0, 1]$ and $(x, u) \in \mathbb{R}^d \times U$:

$$|f^\delta - f|(x, u) \leq K\delta, \tag{5.9}$$

and

$$|f^\delta - f^{\delta'}|(x, u) = |\delta u^2 - \delta' u^2| \leq K|\delta - \delta'|. \tag{5.10}$$

Remark 5.2.1. It is clear that the function f^δ is Lipschitz continuous in x and has the same Lipschitz constant as function f, i.e., there is some positive constant C such that for all $x, x' \in \mathbb{R}^d$ and $u \in U$

$$|f^\delta(x, u) - f^\delta(x', u)| \leq C|x - x'|. \tag{5.11}$$

For any initial condition $(t, x) \in [0, T] \times \mathbb{R}^d$, any admissible control $u \in \mathcal{U}_\nu[t, T]$ and any $\delta \in (0, 1]$, the corresponding system of stochastic differential equations takes the form

$$\begin{cases} dX_s^{t,x,u,\delta} = b(X_s^{t,x,u,\delta}, u_s)ds + \sigma^\delta(X_s^{t,x,u,\delta})dW_s, \\ dY_s^{t,x,u,\delta} = -f^\delta(X_s^{t,x,u,\delta}, u_s)ds + Z_s^{t,x,u,\delta}dW_s, \quad s \in [t, T], \\ X_t^{t,x,u,\delta} = x, \quad Y_T^{t,x,u,\delta} = k(X_T^{t,x,u,\delta}). \end{cases} \tag{5.12}$$

For later use we will show with the following Lemma the smooth dependence of the solution of the forward equation on the choice of $\delta \in (0,1]$.

Lemma 5.2.2. *Let us consider the stochastic forward equation in (5.12). Then, there exists some constant* $K > 0$ *such that for any* $(t,x) \in [0,T] \times \mathbb{R}^d$*,* $u \in U$ *and* $\delta, \delta' \in (0,1]$

$$E\left[\sup_{s\in[t,T]} |X_s^{t,x,u,\delta} - X_s^{t,x,u,\delta'}|^2\right] \leq K|\delta - \delta'|^2.$$

Proof. For any $(t,x) \in [0,T] \times \mathbb{R}^d$, $u \in U$ and $\delta, \delta' \in (0,1]$ we have

$$\begin{aligned}
& |X_s^{t,x,u,\delta} - X_s^{t,x,u,\delta'}|^2 \\
\leq\ & 2\left(\sup_{s\in[t,T]}\left|\int_t^s (b(X_s^{t,x,u,\delta}, u_s) - b(X_s^{t,x,u,\delta'}, u_s))d\tau\right|^2\right. \\
+\ & \left.\sup_{s\in[t,T]}\left|\int_t^s (\sigma^\delta(X_\tau^{t,x,u,\delta}, u_\tau) - \sigma^{\delta'}(X_\tau^{t,x,u,\delta'}, u_\tau))dW_\tau\right|^2\right).
\end{aligned}$$

Let in the following K be some constant that may vary from line to line. Then it follows by using the Burkholder-Davis-Gundy inequality, the Lipschitz continuity of b and σ^δ, $\delta \in (0,1]$, and inequality (5.7) that for any $(t,x) \in [0,T] \times \mathbb{R}^d$, $u \in \mathcal{U}_\nu[t,T]$ and $\delta, \delta' \in (0,1]$

$$\begin{aligned}
& E\left[\sup_{s\in[t,T]} |X_s^{t,x,u,\delta} - X_s^{t,x,u,\delta'}|^2\right] \\
\leq\ & K\left(E\left[\sup_{s\in[t,T]}\left(\int_t^s \left|b(X_\tau^{t,x,u,\delta}, u_\tau) - b(X_\tau^{t,x,u,\delta'}, u_\tau)\right|^2 d\tau\right)\right]\right. \\
+\ & \left.E\left[\int_t^T \left\|\sigma^\delta(X_\tau^{t,x,u,\delta}, u_\tau) - \sigma^{\delta'}(X_\tau^{t,x,u,\delta'}, u_\tau)\right\|^2 d\tau\right]\right) \\
\leq\ & K\left(E\left[\int_t^T \left|b(X_\tau^{t,x,u,\delta}, u_\tau) - b(X_\tau^{t,x,u,\delta'}, u_\tau)\right|^2 d\tau\right]\right. \\
+\ & E\left[\int_t^T \left\|\sigma^\delta(X_\tau^{t,x,u,\delta}) - \sigma^{\delta'}(X_\tau^{t,x,u,\delta})\right\|^2 d\tau\right] \\
+\ & \left.E\left[\int_t^T \left\|\sigma^{\delta'}(X_\tau^{t,x,u,\delta}) - \sigma^{\delta'}(X_\tau^{t,x,u,\delta'})\right\|^2 d\tau\right]\right) \\
\leq\ & K\left(E\left[\int_t^T \left|X_\tau^{t,x,u,\delta} - X_\tau^{t,x,u,\delta'}\right|^2 d\tau\right] + |\delta - \delta'|^2\right).
\end{aligned}$$

Applying Gronwall's lemma gives the desired result.

□

In analogy to the original stochastic control problem, we define the following cost functional for the approximating control problem:

$$J^\delta(t,x,u) \equiv Y_t^{t,x,u,\delta} = E\left[\int_t^T f^\delta(X_s^{t,x,u,\delta},u_s)ds + k(X_T^{t,x,u,\delta})|\mathcal{F}_t\right], \quad (t,x)\in[0,T]\times\mathbb{R}^d, u\in\mathcal{U}_\nu[t,T], \delta\in(0,1]. \tag{5.13}$$

Consider now the corresponding HJB equation for $(t,x)\in[0,T]\times\mathbb{R}^d$ and $\delta\in(0,1]$.

$$\begin{cases} -V_t^\delta(t,x) + \sup_{v\in U} G^\delta\left(x,v,-\nabla_x V^\delta(t,x),-\nabla_{xx}V^\delta(t,x)\right) = 0, \\ V^\delta(T,x) = k(x), \end{cases} \tag{5.14}$$

with the Hamiltonian

$$G^\delta(x,v,p,P) \equiv \frac{1}{2}tr\left(P(\sigma\sigma^*(x)+\delta^2 I_d)\right) + \langle p, b(x,v)\rangle - f^\delta(x,v),$$

for $(x,v,p,P)\in\mathbb{R}^d\times U\times\mathbb{R}^d\times\mathbb{S}^d$.
Note that since the matrix σ^δ is invertible, the drift function of the forward SDE in (5.12) can be transformed as follows for any $x\in\mathbb{R}^d$ and $u\in U$:

$$\begin{aligned} b(x,u) &= b^1(x)+b^2(x)u \\ &= b^1(x)+\sigma^\delta(x)\sigma^\delta(x)^{-1}b^2(x)u \\ &= b^1(x)+\sigma^\delta(x)\theta^\delta(x,u), \end{aligned}$$

where $\theta^\delta(x,u)\equiv\sigma^\delta(x)^{-1}b^2(x)u$ for any $\delta\in(0,1]$. By Assumption 5.1.1, we obtain that $\theta^\delta\in C^1(\mathbb{R}^d\times U)$ with θ^δ and $\nabla_x\theta^\delta$ bounded. Therefore, the coefficients of system (5.12) fulfill the conditions of Theorem 3.3.5 which says that the unique viscosity solution V^δ of the above equation belongs to $C_p^{1,2}([0,T]\times\mathbb{R}^d)$. In addition, Theorem 3.3.6 allows us to find a measurable function $v^\delta:[0,T]\times\mathbb{R}^d\to U$ such that for all $(t,x)\in[0,T]\times\mathbb{R}^d$

$$G^\delta\left(x,v^\delta(t,x),-\nabla_x V^\delta(t,x),-\nabla_{xx}V^\delta(t,x)\right) = \sup_{v\in U} G^\delta\left(x,v,-\nabla_x V^\delta(t,x),-\nabla_{xx}V^\delta(t,x)\right).$$

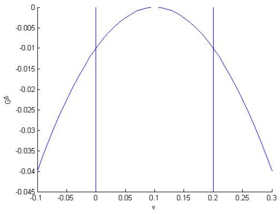
By using the knowledge that the functions b and f are linear with respect to u, we can give more details about the structure of v^δ, since for any $(t,x)\in[0,T]\times\mathbb{R}^d$ and $\delta\in(0,1]$, v^δ maximizes

$$\begin{aligned} & G^\delta\left(x,v,-\nabla_x V^\delta(t,x),-\nabla_{xx}V^\delta(t,x)\right) \\ =\ & -\frac{1}{2}tr\left(\nabla_{xx}V^\delta(t,x)((\sigma\sigma^*)(x)+\delta^2 I_d)\right) - (b^1(x)+b^2(x)v)\nabla_x V^\delta(t,x) - f^1(x) \\ -\ & f^2(x)v - \delta v^2. \end{aligned}$$

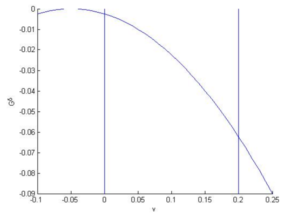
It is easy to check that the following function $v:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ maximizes the above expression if we ignore the bounds C_1 and C_2 of the control process.

$$v(t,x) = -\frac{b^2(x)\nabla_x V^\delta(t,x) + f^2(x)}{2\delta}, \quad (t,x) \in [0,T] \times \mathbb{R}^d, \delta \in (0,1].$$

Next, we take into account the bounds of the control process. Note that for any fixed $\delta \in (0,1]$ the function G^δ is a polynomial of degree 2 in v and therefore takes a parabolic form. Let us fix for a graphical example some $(t,x) \in [0,T] \times \mathbb{R}^d$ and let us assume that $C_1 = 0$ and $C_2 = 0.2$. Consider first the case where the maximizing input value v of the function G^δ lies between the bounds C_1 and C_2.



In this case the restricting bounds on the control variable C_1 and C_2 do not influence the maximizing input value of G^δ.
Let us assume next that the maximizing input value of G^δ is smaller than C_1, i.e., $-\frac{b^2(x)\nabla_x V^\delta(t,x)+f^2(x)}{2\delta} < C_1$.



Since the function is strictly decreasing for $v \in [C_1, C_2]$, we obtain in this case that C_1 is the maximizing input value. Similarly, C_2 is the maximizing input value if $-\frac{b^2(x)\nabla_x V^\delta(t,x)+f^2(x)}{2\delta} > C_2$. Consequently, by taking into account the bounds on the control variable, we obtain $v^\delta : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, which for any $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\delta \in (0,1]$ takes the form

$$v^\delta(t,x) = C_1 \vee \left(-\frac{b^2(x)\nabla_x V^\delta(t,x) + f^2(x)}{2\delta} \right) \wedge C_2. \tag{5.15}$$

Let us now consider the following stochastic differential equation for any $(t,x) \in [0,T] \times$

$\mathbb{R}^d$ and $\delta \in (0,1]$:

$$\begin{aligned} dX_s^{t,x,\delta} &= b(X_s^{t,x,\delta}, v^\delta(s, X_s^{t,x,\delta}))ds + (\sigma(X_s^{t,x,\delta}) + \delta I_d)dW_s \\ &= \tilde{b}^\delta(s, X_s^{t,x,\delta})ds + \sigma^\delta(X_s^{t,x,\delta})dW_s, \quad s \in [t,T], \\ X_t^{t,x,\delta} &= x, \end{aligned} \tag{5.16}$$

where the function $\tilde{b}^\delta : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is defined as $\tilde{b}^\delta(s,x) \equiv b(x, v^\delta(s,x))$.
Note that for any $\delta \in (0,1]$, the function v^δ is Lipschitz continuous in x. It follows that the function $\tilde{b}^\delta$ is Lipschitz continuous in x as well. To see this, consider some $s \in [0,T]$ and $x_1, x_2 \in \mathbb{R}^d$. Then for $\delta \in (0,1]$

$$\begin{aligned} & |\tilde{b}^\delta(s,x_1) - \tilde{b}^\delta(s,x_2)| \\ = \; & |b(x_1, v^\delta(s,x_1)) - b(x_2, v^\delta(s,x_2))| \\ = \; & |b^1(x_1) + v^\delta(s,x_1)b^2(x_1) - b^1(x_2) - v^\delta(s,x_2)b^2(x_2)| \\ \leq \; & |b^1(x_1) - b^1(x_2)| + |v^\delta(s,x_1)b^2(x_1) - v^\delta(s,x_2)b^2(x_2)| \\ \leq \; & |b^1(x_1) - b^1(x_2)| + |v^\delta(s,x_1)b^2(x_1) - v^\delta(s,x_2)b^2(x_1)| \\ + \; & |v^\delta(s,x_2)b^2(x_1) - v^\delta(s,x_2)b^2(x_2)| \\ = \; & |b^1(x_1) - b^1(x_2)| + |b^2(x_1)| \cdot |v^\delta(s,x_1) - v^\delta(s,x_2)| + |v^\delta(s,x_2)| \cdot |b^2(x_1) - b^2(x_2)|. \end{aligned}$$

From the boundedness of the functions v^δ and b^2 and the Lipschitz continuity of the functions b^1, b^2 and v^δ the Lipschitz continuity of the function $\tilde{b}^\delta$ follows. In addition, it is easy to check that $\tilde{b}^\delta$ also fulfills the linear growth condition. Consequently, since the function σ^δ is Lipschitz continuous and bounded, it follows from Theorem 2.4.4 that for any $(t,x) \in [0,T] \times \mathbb{R}$ and $\delta \in (0,1]$ there exists a unique strong solution $X^{t,x,\delta}$ to the stochastic differential equation (5.16). The corresponding control process for $(t,x) \in [0,T] \times \mathbb{R}$ and $\delta \in (0,1]$ is defined as

$$u_s^{t,x,\delta} := v^\delta(s, X_s^{t,x,\delta}), \quad s \in [t,T]. \tag{5.17}$$

Remark 5.2.3. Note that in contrast to us Buckdahn et al. [2010] do not have further information about the structure of the coefficients. Therefore, they can prove only the existence of a weak solution to the forward equation. Consequently, they develop their arguments always on some reference stochastic system $\nu_\delta = (\Omega^\delta, \mathcal{F}^\delta, \left(\mathcal{F}_s^\delta\right)_{s \geq 0}, P^\delta, W^\delta)$ with index δ corresponding to the process $X^{t,x,\delta}$.

The following proposition identifies the solution V^δ of the HJB equation (5.14) as the value function of our approximating control problem, gives an estimate of the distance between the value function V^δ and that of our original control problem and shows that $\nabla_x V^\delta$ is uniformly bounded in δ.

Proposition 5.2.4. *Under Assumption 5.1.1 the following holds.*
1) For all $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\delta \in (0,1]$

$$J^\delta(t,x,u^{t,x,\delta}) = V^\delta(t,x) = \inf_{u \in \mathcal{U}_\nu[t,T]} J^\delta(t,x,u),$$

with $u_s^{t,x,\delta} = v^\delta(s, X_s^{t,x,\delta})$, $s \in [t,T]$.
2) There is some constant C such that for all $(t,x),(t',x') \in [0,T] \times \mathbb{R}^d$ and for all $\delta, \delta' \in (0,1]$

$$|V^{\delta'}(t',x') - V^\delta(t,x)| \leq C|\delta - \delta'| + C(1+|x|+|x'|)|t-t'|^{1/2} + C|x-x'|. \tag{5.18}$$

3) The function V^δ, $\delta \in (0,1]$, converges uniformly to the value function V of the original control problem as $\delta \to 0$.
4) For some constant C the following holds true for all $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\delta \in (0,1]$:

$$\nabla_x V^\delta(t,x) \leq C.$$

Proof. The proof of part 1) is equal to the proof of Proposition 2 in Buckdahn et al. [2010] with a slightly different notation and proceeds as follows.
For $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\delta \in (0,1]$ it is well known that $V^\delta(t,x) = \inf_{u \in \mathcal{U}_\nu[t,T]} J^\delta(t,x,u)$. Therefore it only remains to show that $J^\delta(t,x,u^{t,x,\delta}) = V^\delta(t,x)$. For this we observe that from the uniqueness of the solution of the controlled forward equation with control process $u^{t,x,\delta}$ it follows that $X^{t,x,u^{t,x,\delta},\delta} = X^{t,x,\delta}$ for all $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\delta \in (0,1]$, where $X^{t,x,u^{t,x,\delta},\delta}$ is the solution of the forward process in (5.12) and $X^{t,x,\delta}$ is the solution of SDE (5.16). Moreover, for $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\delta \in (0,1]$ let

$$Y_s^{t,x,\delta} = V^\delta(s, X_s^{t,x,\delta}) \text{ and } Z_s^{t,x,\delta} = \nabla_x V^\delta(s, X_s^{t,x,\delta}) \sigma^\delta(X_s^{t,x,\delta}), \quad s \in [t,T],$$

and notice that $(Y_s^{t,x,\delta}, Z_s^{t,x,\delta})$ belongs to $\mathcal{S}_\nu^2([t,T];\mathbb{R}) \times \mathcal{H}_\nu^2([t,T];\mathbb{R}^d)$.
The aim of the following arguments is to show that $Y_s^{t,x,\delta}$ satisfies the BSDE in (5.12) for all $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\delta \in (0,1]$. Since $V^\delta \in C^{1,2}([0,T] \times \mathbb{R}^d)$, we can apply Itô's formula to obtain for any $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\delta \in (0,1]$

$$\begin{aligned}
& dV^\delta(s, X_s^{t,x,\delta}) \\
= \; & V_s^\delta(s, X_s^{t,x,\delta}) ds + \nabla_x V^\delta(s, X_s^{t,x,\delta}) dX_s^{t,x,\delta} + \frac{1}{2} tr \left((\sigma^\delta \sigma^{\delta *})(X_s^{t,x,\delta}) \nabla_{xx} V^\delta(s, X_s^{t,x,\delta}) \right) ds \\
= \; & \Big(V_s^\delta(s, X_s^{t,x,\delta}) + \nabla_x V^\delta(s, X_s^{t,x,\delta}) b(X_s^{t,x,\delta}, v^\delta(s, X_s^{t,x,\delta})) + \\
+ \; & \frac{1}{2} tr \left((\sigma^\delta \sigma^{\delta *})(X_s^{t,x,\delta}) \nabla_{xx} V^\delta(s, X_s^{t,x,\delta}) \right) \Big) ds + \nabla_x V^\delta(s, X_s^{t,x,\delta}) \sigma^\delta(X_s^{t,x,\delta}) dW_s \\
= \; & -f^\delta(X_s^{t,x,\delta}, v^\delta(s, X_s^{t,x,\delta})) ds + Z_s^{t,x,\delta} dW_s, \quad s \in [t,T].
\end{aligned} \tag{5.19}$$

For the last line we use the HJB equation (5.14) as well as the optimality of the feedback control v^δ defined in equation (5.15), i.e., for all $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\delta \in (0,1]$

$$0 = -V_s^\delta(s, X_s^{t,x,\delta}) + G^\delta\left(x, v^\delta(s, X_s^{t,x,\delta}), -\nabla_x V^\delta(s, X_s^{t,x,\delta}), -\nabla_{xx} V^\delta(s, X_s^{t,x,\delta})\right),$$

which is equivalent to

$$\begin{aligned} & -f^\delta(X_s^{t,x,\delta}, v^\delta(s, X_s^{t,x,\delta})) \\ = \;& V_s^\delta(s, X_s^{t,x,\delta}) + \nabla_x V^\delta(s, X_s^{t,x,\delta}) b(X_s^{t,x,\delta}, v^\delta(s, X_s^{t,x,\delta})) \\ + \;& \frac{1}{2} tr\left((\sigma^\delta \sigma^{\delta *})(X_s^{t,x,\delta}) \nabla_{xx} V^\delta(s, X_s^{t,x,\delta})\right). \end{aligned}$$

Since in addition

$$Y_T^{t,x,\delta} = V^\delta(T, X_T^{t,x,\delta}) = k(X_T^{t,x,\delta}),$$

it follows from (5.19) that for all $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\delta \in (0,1]$ $(Y_s^{t,x,\delta}, Z_s^{t,x,\delta})$ satisfies the BSDE in (5.12) for $u = u^{t,x,\delta}$.
From the uniqueness of the solution to BSDE (5.12) it follows for $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\delta \in (0,1]$ that $(Y_s^{t,x,u^{t,x,\delta},\delta}, Z_s^{t,x,u^{t,x,\delta},\delta})_{s \in [t,T]} = (Y_s^{t,x,\delta}, Z_s^{t,x,\delta})_{s \in [t,T]}$. Thus, from the definition of $Y^{t,x,\delta}$ we get, in particular, that for $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\delta \in (0,1]$

$$J^\delta(t,x,u^{t,x,\delta}) = Y_t^{t,x,u^{t,x,\delta},\delta} = Y_t^{t,x,\delta} = V^\delta(t,x). \tag{5.20}$$

2) The aim is to show the continuous dependence of $V^\delta(t,x)$ on the initial values $t \in [0,T]$, $x \in \mathbb{R}^d$ and the value $\delta \in (0,1]$. The proof again relies very closely on the proof of Proposition 2 in Buckdahn et al. [2010].
Let us fix for what follows some arbitrary $\delta \in (0,1]$ and $(t,x) \in [0,T] \times \mathbb{R}^d$ and consider the corresponding control process $u^{t,x,\delta}$ defined in (5.17). We extend this process onto the whole interval by setting $u_s^{t,x,\delta} = u_t^{t,x,\delta}$ for $s \in [0,t]$. For $\delta' \in (0,1]$ and $(t',x') \in [0,T] \times \mathbb{R}^d$ we let $X^{t',x',u^{t,x,\delta},\delta'} \in \mathcal{S}_\nu^2([t',T];\mathbb{R}^d)$ denote the unique solution of the forward equation

$$\begin{cases} dX_s^{t',x',u^{t,x,\delta},\delta'} = b(X_s^{t',x',u^{t,x,\delta},\delta'}, u_s^{t,x,\delta})ds + \sigma^{\delta'}(X_s^{t',x',u^{t,x,\delta},\delta'})dW_s, \quad s \in [t',T], \\ X_{t'}^{t',x',u^{t,x,\delta},\delta'} = x'. \end{cases} \tag{5.21}$$

We extend this solution process onto the whole interval $[0,T]$ by setting $X_s^{t',x',u^{t,x,\delta},\delta'} = x'$, for $s < t'$. Then, by putting for $s \in [t',T]$

$$\begin{aligned} \tilde{f}_s^{t',x',u^{t,x,\delta},\delta'} \;=\; & -\Bigg(V_s^{\delta'}(s, X_s^{t',x',u^{t,x,\delta},\delta'}) + \frac{1}{2} tr\left(\left((\sigma\sigma^*)(X_s^{t',x',u^{t,x,\delta},\delta'}) + \delta'^2 I_d\right) \times\right. \\ & \left.\nabla_{xx} V^\delta(s, X_s^{t',x',u^{t,x,\delta},\delta'})\right) + b(X_s^{t',x',u^{t,x,\delta},\delta'}, u_s^{t,x,\delta}) \nabla_x V^{\delta'}(s, X_s^{t',x',u^{t,x,\delta},\delta'})\Bigg), \end{aligned} \tag{5.22}$$

we define a stochastic process in $\mathcal{H}_\nu^2([t',T];\mathbb{R})$. Applying Itô's formula to $V^{\delta'}(s, X_s^{t',x',u^{t,x,\delta},\delta'})$ yields for $s \in [t',T]$

$$dV^{\delta'}(s, X_s^{t',x',u^{t,x,\delta},\delta'})$$

$$
\begin{aligned}
&= V_s^{\delta'}(s, X_s^{t',x',u^{t,x,\delta},\delta'})ds + \nabla_x V^{\delta'}(s, X_s^{t',x',u^{t,x,\delta},\delta'})dX_s^{t',x',u^{t,x,\delta},\delta'} + \\
&\quad \frac{1}{2}tr\left(\left((\sigma\sigma^*)(X_s^{t',x',u^{t,x,\delta},\delta'}) + \delta'^2 I_d\right)\nabla_{xx}V^{\delta'}(s, X_s^{t',x',u^{t,x,\delta},\delta'})\right)ds \\
&= \left(V_s^{\delta'}(s, X_s^{t',x',u^{t,x,\delta},\delta'}) + \frac{1}{2}tr\left(\left((\sigma\sigma^*)(X_s^{t',x',u^{t,x,\delta},\delta'}) + \delta'^2 I_d\right)\nabla_{xx}V^{\delta'}(s, X_s^{t',x',u^{t,x,\delta},\delta'})\right)\right. \\
&+ \left. b(X_s^{t',x',u^{t,x,\delta},\delta'}, u_s^{t,x,\delta})\nabla_x V^{\delta'}(s, X_s^{t',x',u^{t,x,\delta},\delta'})\right)ds \\
&+ \nabla_x V^{\delta'}(s, X_s^{t',x',u^{t,x,\delta},\delta'})\sigma^{\delta'}(X_s^{t',x',u^{t,x,\delta},\delta'})dW_s.
\end{aligned}
$$

We see that

$$\tilde{Y}_s^{t',x',u^{t,x,\delta},\delta'} = V^{\delta'}(s, X_s^{t',x',u^{t,x,\delta},\delta'})$$

and

$$\tilde{Z}_s^{t',x',u^{t,x,\delta},\delta'} = \nabla_x V^{\delta'}(s, X_s^{t',x',u^{t,x,\delta},\delta'})\sigma^{\delta}(X_s^{t',x',u^{t,x,\delta},\delta'})$$

is the unique solution of the BSDE

$$
\begin{cases}
d\tilde{Y}_s^{t',x',u^{t,x,\delta},\delta'} = -\tilde{f}_s^{t',x',u^{t,x,\delta},\delta'}ds + \tilde{Z}_s^{t',x',u^{t,x,\delta},\delta'}dW_s, \quad s \in [t', T], \\
\tilde{Y}_T^{t',x',u^{t,x,\delta},\delta'} = k(X_T^{t',x',u^{t,x,\delta},\delta'}), \\
(\tilde{Y}_s^{t',x',u^{t,x,\delta},\delta'}, \tilde{Z}_s^{t',x',u^{t,x,\delta},\delta'}) \in \mathcal{S}_\nu^2([t', T]; \mathbb{R}) \times \mathcal{H}_\nu^2([t', T]; \mathbb{R}^d).
\end{cases}
\tag{5.23}
$$

We want to compare the first component of the solution of (5.23) with that of the BSDE

$$
\begin{cases}
dY_s^{t',x',u^{t,x,\delta},\delta'} = -f^{\delta'}(X_s^{t',x',u^{t,x,\delta},\delta'}, u_s^{t,x,\delta})ds + Z_s^{t',x',u^{t,x,\delta},\delta'}dW_s, \quad s \in [t', T], \\
Y_T^{t',x',u^{t,x,\delta},\delta'} = k(X_T^{t',x',u^{t,x,\delta},\delta'}), \\
(Y_s^{t',x',u^{t,x,\delta},\delta'}, Z_s^{t',x',u^{t,x,\delta},\delta'}) \in \mathcal{S}_\nu^2([t', T]; \mathbb{R}) \times \mathcal{H}_\nu^2([t', T]; \mathbb{R}^d).
\end{cases}
\tag{5.24}
$$

From the Hamilton-Jacobi-Bellman equation with classical solution $V^{\delta'}$ we have

$$
\begin{aligned}
0 &= -V_s^{\delta'}(s, X_s^{t',x',u^{t,x,\delta},\delta'}) \\
&- \frac{1}{2}tr\left(\left((\sigma\sigma^*)(X_s^{t',x',u^{t,x,\delta},\delta'}) + \delta'^2 I_d\right)\nabla_{xx}V^{\delta'}(s, X_s^{t',x',u^{t,x,\delta},\delta'})\right) \\
&+ \sup_{v\in U}\left(-b(X_s^{t',x',u^{t,x,\delta},\delta'}, v)\nabla_x V^{\delta'}(s, X_s^{t',x',u^{t,x,\delta},\delta'}) - f^{\delta'}(X_s^{t',x',u^{t,x,\delta},\delta'}, v)\right).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
0 &\geq -V_s^{\delta'}(s, X_s^{t',x',u^{t,x,\delta},\delta'}) \\
&- \frac{1}{2}tr\left(\left((\sigma\sigma^*)(X_s^{t',x',u^{t,x,\delta},\delta'}) + \delta'^2 I_d\right)\nabla_{xx}V^{\delta'}(s, X_s^{t',x',u^{t,x,\delta},\delta'})\right) \\
&- b(X_s^{t',x',u^{t,x,\delta},\delta'}, u^{t,x,\delta})\nabla_x V^{\delta'}(s, X_s^{t',x',u^{t,x,\delta},\delta'}) - f^{\delta'}(X_s^{t',x',u^{t,x,\delta},\delta'}, u^{t,x,\delta}).
\end{aligned}
$$

By using (5.22), the last inequality can be written as

$$\tilde{f}_s^{t',x',u^{t,x,\delta},\delta'} \leq f^{\delta'}\left(X_s^{t',x',u^{t,x,\delta},\delta'}, u_s^{t,x,\delta}\right), \quad s \in [t', T].$$

Then the Comparison Theorem for BSDEs (see Theorem 2.5.5) yields $\tilde{Y}_s^{t',x',u^{t,x,\delta},\delta'} \leq Y_s^{t',x',u^{t,x,\delta},\delta'}$, $s \in [t',T]$, P-a.s., and consequently, due to 1),

$$\begin{aligned} V^{\delta'}(t',x') - V^{\delta}(t,x) = \tilde{Y}_{t'}^{t',x',u^{t,x,\delta},\delta'} - Y_t^{t,x,\delta} \leq Y_{t'}^{t',x',u^{t,x,\delta},\delta'} - Y_t^{t,x,\delta}, \\ (t,x),(t',x') \in [0,T]\times\mathbb{R}^d, \delta,\delta' \in (0,1], P-a.s. \end{aligned}$$

For $(t,x),(t',x') \in [0,T]\times\mathbb{R}^d$, $\delta,\delta' \in (0,1]$ applying Itô's formula to $\sup_{s\in[0,T]} |Y_s^{t',x',u^{t,x,\delta},\delta'} - Y_s^{t,x,\delta}|^2$ and taking the expectation yields with the usual convention $Y_s^{t',x',u^{t,x,\delta},\delta'} = Y_{t'}^{t',x',u^{t,x,\delta},\delta'} = 0$ for all $s < t'$,

$$\begin{aligned} & E\left[\sup_{s\in[0,T]} |Y_s^{t',x',u^{t,x,\delta},\delta'} - Y_s^{t,x,\delta}|^2 + \int_0^T |Z_s^{t',x',u^{t,x,\delta},\delta'} - Z_s^{t,x,\delta}|^2 ds\right] \\ = \; & E\left[|k(X_T^{t',x',u^{t,x,\delta},\delta'}) - k(X_T^{t,x,\delta})|^2\right] \\ + \; & 2E\left[\sup_{s\in[0,T]}\left|\int_s^T \left\langle Y_r^{t',x',u^{t,x,\delta},\delta'} - Y_r^{t,x,\delta}, f^{\delta'}(X_r^{t',x',u^{t,x,\delta},\delta'}, u_r^{t,x,\delta}) - f^{\delta}(X_r^{t,x,\delta}, u_r^{t,x,\delta})\right\rangle dr\right|\right] \\ + \; & 2E\left[\sup_{s\in[0,T]}\left|\int_s^T \left\langle Y_r^{t',x',u^{t,x,\delta},\delta'} - Y_r^{t,x,\delta}, \left(Z_r^{t',x',u^{t,x,\delta},\delta'} - Z_r^{t,x,\delta}\right) dW_r\right\rangle\right|\right]. \end{aligned}$$

By the inequalities of Young and Burkholder-Davis-Gundy there exists a constant γ such that for any $\epsilon > 0$

$$\begin{aligned} & E\left[\sup_{s\in[0,T]} |Y_s^{t',x',u^{t,x,\delta},\delta'} - Y_s^{t,x,\delta}|^2 + \int_0^T |Z_s^{t',x',u^{t,x,\delta},\delta'} - Z_s^{t,x,\delta}|^2 ds\right] \\ \leq \; & E\left[|k(X_T^{t',x',u^{t,x,\delta},\delta'}) - k(X_T^{t,x,\delta})|^2\right] + \frac{1}{\epsilon}E\left[\int_0^T |Y_s^{t',x',u^{t,x,\delta},\delta'} - Y_s^{t,x,\delta}|^2 ds\right] \\ + \; & \epsilon E\left[\int_0^T |f^{\delta'}(X_s^{t',x',u^{t,x,\delta},\delta'}, u_s^{t,x,\delta}) - f^{\delta}(X_s^{t,x,\delta}, u_s^{t,x,\delta})|^2 ds\right] \\ + \; & 2\gamma E\left[\int_0^T |Y_s^{t',x',u^{t,x,\delta},\delta'} - Y_s^{t,x,\delta}|^2 \cdot |Z_s^{t',x',u^{t,x,\delta},\delta'} - Z_s^{t,x,\delta}|^2 ds\right] \\ \leq \; & E\left[|k(X_T^{t',x',u^{t,x,\delta},\delta'}) - k(X_T^{t,x,\delta})|^2\right] + \frac{1}{\epsilon}E\left[\int_0^T |Y_s^{t',x',u^{t,x,\delta},\delta'} - Y_s^{t,x,\delta}|^2 ds\right] \\ + \; & \epsilon E\left[\int_0^T |f^{\delta'}(X_s^{t',x',u^{t,x,\delta},\delta'}, u_s^{t,x,\delta}) - f^{\delta}(X_s^{t,x,\delta}, u_s^{t,x,\delta})|^2 ds\right] \\ + \; & \frac{\gamma}{\epsilon}E\left[\int_0^T |Y_s^{t',x',u^{t,x,\delta},\delta'} - Y_s^{t,x,\delta}|^2 ds\right] + \gamma\epsilon E\left[\int_0^T |Z_s^{t',x',u^{t,x,\delta},\delta'} - Z_s^{t,x,\delta}|^2 ds\right]. \end{aligned}$$

By rearranging and using Gronwall's lemma we obtain a constant K such that

$$\begin{aligned}
& E\left[\sup_{s\in[0,T]}|Y_s^{t',x',u^{t,x,\delta},\delta'} - Y_s^{t,x,\delta}|^2 + \int_0^T |Z_s^{t',x',u^{t,x,\delta},\delta'} - Z_s^{t,x,\delta}|^2 ds\right] \\
\leq\ & K\left(E\left[|k(X_T^{t',x',u^{t,x,\delta},\delta'}) - k(X_T^{t,x,\delta})|^2\right]\right. \\
+\ & \left. E\left[\int_0^T |f^{\delta'}(X_s^{t',x',u^{t,x,\delta},\delta'}, u_s^{t,x,\delta}) - f^{\delta}(X_s^{t,x,\delta}, u_s^{t,x,\delta})|^2 ds\right]\right) \\
\leq\ & K\left(E\left[|k(X_T^{t',x',u^{t,x,\delta},\delta'}) - k(X_T^{t,x,\delta})|^2\right]\right. \\
+\ & E\left[\int_0^T |f^{\delta'}(X_s^{t',x',u^{t,x,\delta},\delta'}, u_s^{t,x,\delta}) - f^{\delta'}(X_s^{t,x,\delta}, u_s^{t,x,\delta})|^2 ds\right] \\
+\ & \left. E\left[\int_0^T |f^{\delta'}(X_s^{t,x,\delta}, u_s^{t,x,\delta}) - f^{\delta}(X_s^{t,x,\delta}, u_s^{t,x,\delta})|^2 ds\right]\right).
\end{aligned}$$

Using the Lipschitz continuity of the functions k and f^δ, $\delta \in (0,1]$ (see Remark (5.1.2) and Remark (5.2.1)), and using inequality (5.10) yields for some constant C

$$\begin{aligned}
& E\left[\sup_{s\in[0,T]}|Y_s^{t',x',u^{t,x,\delta},\delta'} - Y_s^{t,x,\delta}|^2 + \int_0^T |Z_s^{t',x',u^{t,x,\delta},\delta'} - Z_s^{t,x,\delta}|^2 ds\right] \\
\leq\ & C\left((\delta-\delta')^2 + E\left[\sup_{s\in[0,T]}|X_s^{t',x',u^{t,x,\delta},\delta'} - X_s^{t,x,\delta}|^2\right]\right) \\
\leq\ & C\left((\delta-\delta')^2 + E\left[\sup_{s\in[0,T]}|X_s^{t',x',u^{t,x,\delta},\delta'} - X_s^{t',x',u^{t,x,\delta},\delta}|^2\right]\right. \\
+\ & \left. E\left[\sup_{s\in[0,T]}|X_s^{t',x',u^{t,x,\delta},\delta} - X_s^{t,x,\delta}|^2\right]\right) \\
\leq\ & C(\delta-\delta')^2 + C(1+|x|+|x'|)^2|t-t'| + C|x-x'|^2, \\
& (t,x),(t',x') \in [0,T]\times\mathbb{R}^d, \delta,\delta' \in (0,1].
\end{aligned}$$

Note that for the last inequality we used Lemma 5.2.2 and Theorem 2.4.4.
From the symmetry of the arguments w.r.t. the initial data we get

$$|V^{\delta'}(t',x') - V^{\delta}(t,x)| \leq C|\delta-\delta'| + C(1+|x|+|x'|)|t-t'|^{1/2} + C|x-x'|,$$

which ends the proof of 2).

3) Next, we want to show the uniform convergence of $(V^\delta)_{\delta\in(0,1]}$ as $\delta \to 0$. In Buckdahn et al. [2010] this can be shown quite easily because uniform boundedness of $(V^\delta)_{\delta\in(0,1]}$, follows from the boundedness of the coefficients. Together with the result in 2), which says that for any $(t,x) \in \mathbb{R}^d$ we have $|V^{\delta'}(t,x) - V^{\delta}(t,x)| \leq C|\delta-\delta'|$, the uniform convergence of $(V^\delta)_{\delta\in(0,1]}$ follows. Since in contrast to Buckdahn et al. we do not assume

the coefficients to be bounded, we need additional arguments.
For any $(t,x) \in [0,T] \times \mathbb{R}^d$, $u \in \mathcal{U}_\nu[t,T]$ and $\delta \in (0,1]$ consider the processes $(X_s^{t,x,u})_{s\in[t,T]}$ and $(X_s^{t,x,u,\delta})_{s\in[t,T]}$ which are the solutions of the forward equations in (5.1) and (5.12). For any $(t,x) \in [0,T] \times \mathbb{R}^d$, $u \in \mathcal{U}_\nu[t,T]$ and $\delta \in (0,1]$ we have

$$\begin{aligned}
&|X_s^{t,x,u,\delta} - X_s^{t,x,u}|^2 \\
\leq\ & 2\left(\sup_{s\in[t,T]}\left|\int_t^s (b(X_\tau^{t,x,u,\delta}, u_\tau) - b(X_\tau^{t,x,u}, u_\tau))d\tau\right|^2\right. \\
+\ & \left.\sup_{s\in[t,T]}\left|\int_t^s (\sigma^\delta(X_\tau^{t,x,u,\delta}, u_\tau) - \sigma(X_\tau^{t,x,u}, u_\tau))dW_\tau\right|^2\right).
\end{aligned}$$

Let in the following K be some constant that may vary from line to line. Then it follows by using the Burkholder-Davis-Gundy inequality, the Lipschitz continuity of b and σ and inequality (5.6) that for any $(t,x) \in [0,T] \times \mathbb{R}^d$, $u \in \mathcal{U}_\nu[t,T]$ and $\delta \in (0,1]$

$$\begin{aligned}
& E\left[\sup_{s\in[t,T]} |X_s^{t,x,u,\delta} - X_s^{t,x,u}|^2\right] \\
\leq\ & K\left(E\left[\sup_{s\in[t,T]}\left(\int_t^s \left|b(X_\tau^{t,x,u,\delta}, u_\tau) - b(X_\tau^{t,x,u}, u_\tau)\right| d\tau\right)^2\right]\right. \\
+\ & \left. E\left[\int_t^T \left\|\sigma^\delta(X_\tau^{t,x,u,\delta}) - \sigma(X_\tau^{t,x,u}))\right\|^2 d\tau\right]\right) \\
\leq\ & K\left(E\left[\int_t^T \left|b(X_\tau^{t,x,u,\delta}, u_\tau) - b(X_\tau^{t,x,u}, u_\tau)\right|^2 d\tau\right]\right. \\
+\ & \left. E\left[\int_t^T \left\|\sigma^\delta(X_\tau^{t,x,u,\delta}) - \sigma(X_\tau^{t,x,u,\delta})\right\|^2 d\tau\right] + E\left[\int_t^T \left\|\sigma(X_\tau^{t,x,u,\delta}) - \sigma(X_\tau^{t,x,u})\right\|^2 d\tau\right]\right) \\
\leq\ & K\left(E\left[\int_t^T \left|X_\tau^{t,x,u,\delta} - X_\tau^{t,x,u}\right|^2 d\tau\right] + \delta^2\right).
\end{aligned}$$

Applying Gronwall's lemma gives

$$E\left[\sup_{s\in[t,T]} |X_s^{t,x,u,\delta} - X_s^{t,x,u}|^2\right] \leq K\delta^2.$$

Consequently, by Hölder's inequality we obtain

$$E\left[\sup_{s\in[t,T]} |X_s^{t,x,u,\delta} - X_s^{t,x,u}|\right] \leq K\delta. \tag{5.25}$$

Let in the following C denote the maximum of the Lipschitz constants of the functions f^δ, $\delta \in (0,1]$ and k and let K be again some constant that may vary from line to line. For any $(t,x) \in [0,T] \times \mathbb{R}^d$, $u \in \mathcal{U}_\nu[t,T]$ and $\delta \in (0,1]$ we obtain by using equations (5.2) and (5.13)

$$
\begin{aligned}
& |J(t,x,u) - J^\delta(t,x,u)| \\
\leq \ & E\left[\int_t^T |f(X_s^{t,x,u}, u_s) - f^\delta(X_s^{t,x,u,\delta}, u_s)|ds + |k(X_T^{t,x,u}) - k(X_T^{t,x,u,\delta})|\right] \\
\leq \ & E\left[\int_t^T |f(X_s^{t,x,u}, u_s) - f^\delta(X_s^{t,x,u}, u_s)|ds + \int_t^T |f^\delta(X_s^{t,x,u}, u_s) - f^\delta(X_s^{t,x,u,\delta}, u_s)|ds \right. \\
+ \ & \left. C|X_T^{t,x,u} - X_T^{t,x,u,\delta}|\right] \\
\leq \ & E\left[K\delta + C\int_t^T |X_s^{t,x,u} - X_s^{t,x,u,\delta}|ds + C|X_T^{t,x,u} - X_T^{t,x,u,\delta}|\right] \\
\leq \ & KE\left[\delta + \sup_{s\in[t,T]} |X_s^{t,x,u} - X_s^{t,x,u,\delta}|\right] \\
\leq \ & K\delta.
\end{aligned}
$$

Note that in the last line we use (5.25), and in the third line from below we use (5.9). It follows that

$$\lim_{\delta\to 0} |J^\delta(t,x,u) - J(t,x,u)| = 0,$$

uniformly in $(t,x) \in [0,T] \times \mathbb{R}^d$ and $u \in \mathcal{U}[t,T]$. Consequently,

$$\lim_{\delta\to 0} V^\delta(t,x) = V^0(t,x),$$

uniformly in $(t,x,u) \in [0,T] \times \mathbb{R}^d \times U$.
After we showed that $(V^\delta)_{\delta\in(0,1]}$ converges uniformly to V^0 as $\delta \to 0$, it remains to show that V^0 and V coincide, where V denotes the value function of the original problem. For that note that the Hamiltonian H^δ of the approximating control problem converges on compacts to the Hamiltonian of the equation for V. It follows from the stability principle for viscosity solutions (compare Theorem 2.5.13) that V^0 is a viscosity solution of the same equation as V. Thus, since under Assumption 5.1.1 the uniqueness of the viscosity solution holds (compare Theorem 3.3.5), we get that V^0 and V coincide. Consequently, V^δ converges uniformly to V, as $\delta \to 0$.

4) In order to prove the uniform boundedness of $\nabla_x V^\delta(t,x)$ for $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\delta \in (0,1]$, we recall that

$$
\begin{cases}
dX_s^{t,x,\delta} = b(X_s^{t,x,\delta}, u_s^{t,x,\delta})ds + \sigma^\delta(X_s^{t,x,\delta})dW_s, \\
dY_s^{t,x,\delta} = -f^\delta(X_s^{t,x,\delta}, u_s^{t,x,\delta})ds + Z_s^{t,x,\delta}dW_s, \quad s \in [t,T], \\
X_t^{t,x,\delta} = x, \quad Y_T^{t,x,\delta} = k(X_T^{t,x,\delta}).
\end{cases}
\tag{5.26}
$$

From (5.20), we know that

$$V^\delta(t,x) = Y_t^{t,x,\delta}, \quad (t,x) \in [0,T] \times \mathbb{R}^d, \delta \in (0,1],$$

and consequently,

$$\nabla_x V^\delta(t,x) = \nabla_x Y_t^{t,x,\delta}, \quad (t,x) \in [0,T] \times \mathbb{R}^d, \delta \in (0,1]. \tag{5.27}$$

Note that since $\sigma^\delta(x) = \sigma(x) + \delta I_d$ for any $x \in \mathbb{R}^d$, the following holds

$$\nabla_x \sigma^{\delta,j}(x) = \nabla_x \sigma^j(x), \quad j = 1, ...d, \tag{5.28}$$

where σ^j, resp. $\sigma^{\delta,j}$ denotes the j-th column of matrix σ, resp. σ^δ, and $\nabla_x \sigma^j$, resp. $\nabla_x \sigma^{\delta,j}$ denotes the Jacobian of σ^j, resp. $\sigma^{\delta,j}$. In addition, we have $f(x,u) = f^1(x) + f^2(x)u$ and $f^\delta(x,u) = f^1(x) + f^2(x)u + \delta u^2$ for any $(x,u) \in \mathbb{R}^d \times U$ and therefore

$$\nabla_x f^\delta(x,u) = \nabla_x f(x,u), \quad (x,u) \in \mathbb{R}^d \times U. \tag{5.29}$$

From Theorem 2.4.8, we know that formally differentiating $X^{t,x,\delta}$ yields for any $(t,x) \in [0,T] \times \mathbb{R}^d$, $\delta \in (0,1]$ and any $s \in [t,T]$

$$\begin{aligned}
&\nabla_x X_s^{t,x,\delta} \\
= \;& I_d + \int_t^s \nabla_x b(X_\tau^{t,x,\delta}, u_\tau^{t,x,\delta}) \nabla_x X_\tau^{t,x,\delta} d\tau + \int_t^s \sum_{j=1}^d \nabla_x \sigma^{\delta,j}(X_\tau^{t,x,\delta}) \nabla_x X_\tau^{t,x,\delta} dW_\tau^j \\
= \;& I_d + \int_t^s \nabla_x b(X_\tau^{t,x,\delta}, u_\tau^{t,x,\delta}) \nabla_x X_\tau^{t,x,\delta} d\tau + \int_t^s \sum_{j=1}^d \nabla_x \sigma^{j}(X_\tau^{t,x,\delta}) \nabla_x X_\tau^{t,x,\delta} dW_\tau^j.
\end{aligned}$$

For the latter equality we used (5.28).
From Assumption 5.1.1 we know that b and σ are Lipschitz continuous and that their derivatives with respect to x are Lipschitz continuous as well. Therefore, we can use Theorem 2.4.8 to obtain a constant $K_1 \in \mathbb{R}$ such that for all $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\delta \in (0,1]$

$$E\left[\sup_{s\in[t,T]} \left\| \nabla_x X_s^{t,x,\delta} \right\| \right] \leq K_1.$$

By Proposition 2.5.8 we know that differentiating $Y^{t,x,\delta}$ yields for any $(t,x) \in [0,T] \times \mathbb{R}^d$, $\delta \in (0,1]$ and $s \in [t,T]$

$$\begin{aligned}
&\nabla_x Y_s^{t,x,\delta} \\
= \;& k_x(X_T^{t,x,\delta}) \nabla_x X_T^{t,x,\delta} - \int_s^T \nabla_x Z_\tau^{t,x,\delta} dW_\tau + \int_s^T f_x^\delta(X_\tau^{t,x,\delta}, u_\tau^{t,x,\delta}) \nabla_x X_\tau^{t,x,\delta} d\tau \\
= \;& k_x(X_T^{t,x,\delta}) \nabla_x X_T^{t,x,\delta} - \int_s^T \nabla_x Z_\tau^{t,x,\delta} dW_\tau + \int_s^T f_x(X_\tau^{t,x,\delta}, u_\tau^{t,x,\delta}) \nabla_x X_\tau^{t,x,\delta} d\tau.
\end{aligned}$$

For the latter equality we used (5.29).
The uniform boundedness of $\nabla_x V^\delta(t,x)$ is a consequence of the following estimates. In

fact, for all $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\delta \in (0,1]$

$$\begin{aligned}
&|\nabla_x V^\delta(t,x)| = |\nabla_x Y_t^{t,x,\delta}| \\
\leq\ & E\left[|k_x(X_t^{t,x,\delta})| \cdot \left\|\nabla_x X_t^{t,x,\delta}\right\| |\mathcal{F}_t\right] + E\left[\int_t^T |f_x(X_s^{t,x,\delta}, u_s^{t,x,\delta})| \cdot \left\|\nabla_x X_s^{t,x,\delta}\right\| ds|\mathcal{F}_t\right] \\
\leq\ & K_2 E\left[\sup_{t\leq s\leq T} \left\|\nabla_x X_s^{t,x,\delta}\right\| |\mathcal{F}_t\right] + K_3 E\left[\int_t^T \left\|\nabla_x X_s^{t,x,\delta}\right\| ds|\mathcal{F}_t\right] \\
\leq\ & K_2 E\left[\sup_{t\leq s\leq T} \left\|\nabla_x X_s^{t,x,\delta}\right\| |\mathcal{F}_t\right] + TK_3 E\left[\sup_{t\leq s\leq T} \left\|\nabla_x X_s^{t,x,\delta}\right\| |\mathcal{F}_t\right] \\
\leq\ & (K_2 + TK_3)K_1,
\end{aligned}$$

where $K_2, K_3 \in \mathbb{R}$ denote constants which depend on the bounds of k_x and f_x.

□

Remark 5.2.5. From parts 2) and 3) of Proposition 2.2.12 follows the existence of some constant C such that for all $(t,x) \in [0,T] \times \mathbb{R}^d$ and for all $\delta \in (0,1]$

$$|V^\delta(t,x) - V(t,x)| \leq C\delta.$$

5.3. Convergence of the Approximating Control Problems

Recall that in the previous section we studied the following decoupled FBSDE on some d-dimensional Brownian stochastic basis $\nu = (\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P, W)$ for $(t,x) \in [0,T]\times\mathbb{R}^d$ and $\delta \in (0,1]$:

$$\begin{cases} dX_s^{t,x,\delta} = b(X_s^{t,x,\delta}, u_s^{t,x,\delta})ds + \sigma^\delta(X_s^{t,x,\delta})dW_s, \\ dY_s^{t,x,\delta} = -f^\delta(X_s^{t,x,\delta}, u_s^{t,x,\delta})ds + Z_s^{t,x,\delta}dW_s, \quad s \in [t,T], \\ X_t^{t,x,\delta} = x, \quad Y_T^{t,x,\delta} = k(X_T^{t,x,\delta}). \end{cases} \tag{5.30}$$

Here, $u^{t,x,\delta}$ is defined as in (5.17).
We will prove now that for any $(t,x) \in [0,T] \times \mathbb{R}^d$ there exists some initial reference Brownian stochastic basis and a sequence $(\delta_n)_{n\in\mathbb{N}} \subset (0,1]$ with $\lim_{n\to\infty} \delta_n = 0$ such that the sequence of solutions $(X^{t,x,\delta_n}, Y^{t,x,\delta_n})_{n\in\mathbb{N}}$ of the approximating stochastic controlled systems converges in distribution to the solution of the initial stochastic controlled system (5.1). To construct such a convergent sequence we will proceed as in the proof of Theorem 3 in Buckdahn et al. [2010]. This means that for any $(t,x) \in [0,T] \times \mathbb{R}^d$ we will introduce first an auxiliary sequence of systems of **forward** SDEs which has a subsequence whose solutions converge in distribution to a pair $(\bar{X}^{t,x}, \bar{Y}^{t,x})$ associated to a control that is optimal for the original control problem. By means of this convergent subsequence of forward equations we will construct a sequence of solutions $(X^{t,x,\delta_n}, Y^{t,x,\delta_n})_{n\in\mathbb{N}}$ and will show that it has the same limit as the solutions of the subsequence of the auxiliary one. As a consequence we will obtain the existence of an

optimal control of the original control problem.
Let $(\delta_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers converging to 0. The auxiliary sequence of systems of forward SDEs is the following for $(t,x)\in[0,T]\times\mathbb{R}^d$ and $n\in\mathbb{N}$:

$$\begin{cases} dX_s^{t,x,n} = b(X_s^{t,x,n}, u_s^{t,x,\delta_n})ds + \sigma(X_s^{t,x,n})dW_s, \\ dY_s^{t,x,n} = -f(X_s^{t,x,n}, u_s^{t,x,\delta_n})ds + \omega_s^n \sigma(X_s^{t,x,n})dW_s, \quad s\in[t,T], \\ X_t^{t,x,n} = x, \quad Y_t^{t,x,n} = V(t,x), \end{cases} \tag{5.31}$$

with u^{t,x,δ_n} defined as in (5.17) and $\omega_s^{t,x,n} = \nabla_x V^{\delta_n}(s, X_s^{t,x,\delta_n})$. To simplify the notation, we set for all $(x,y,z,v)\in\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times U$

$$\Sigma(x,y,z,v) = \begin{pmatrix} \sigma(x) \\ z^* \end{pmatrix} \quad \text{and} \quad \beta(x,y,z,v) = \begin{pmatrix} b(x,v) \\ -f(x,v) \end{pmatrix}.$$

With the above definitions, (5.31) becomes for $(t,x)\in[0,T]\times\mathbb{R}^d$ and $n\in\mathbb{N}$

$$\begin{cases} d\chi_s^{t,x,n} = \beta(\chi_s^{t,x,n}, r_s^{t,x,n})ds + \Sigma(\chi_s^{t,x,n}, r_s^{t,x,n})dW_s, \quad s\in[t,T], \\ \chi_t^{t,x,n} = \begin{pmatrix} x \\ V(t,x) \end{pmatrix}, \end{cases} \tag{5.32}$$

with $\chi_s^{t,x,n} = \begin{pmatrix} X_s^{t,x,n} \\ Y_s^{t,x,n} \end{pmatrix}$ and $r_s^{t,x,n} = (\omega_s^{t,x,n}\sigma(X_s^{t,x,n}), u_s^{t,x,\delta_n})$ for $s\in[t,T]$ and $n\in\mathbb{N}$.

Remark 5.3.1. From Assumption 5.1.1 and Proposition 5.2.4 it follows that

- β is Lipschitz continuous and has linear growth with respect to x, which means that there exists a constant K_{grow} such that

$$|\beta(x,y,z,v)| \le K_{grow}(1+|x|), \quad (x,y,z,v)\in\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times U,$$

- $\Sigma(\chi_s^{t,x,n}, r_s^{t,x,n})$ is Lipschitz continuous and uniformly bounded for $(t,x,n)\in[0,T]\times\mathbb{R}^d\times\mathbb{N}$ and $s\in[0,T]$.

Proposition 5.3.2. *Let Assumption 5.1.1 (and with this the Roxin condition) be satisfied and let $K>0$. Then for all $x\in\mathbb{R}^d$, there exists a compact set A in $\mathbb{R}^d\times U$, with*

$$\Big\{(\sigma^*(x)\omega, v) | v\in U, \omega\in\mathbb{R}^d \text{ s.t. } |\sigma^*(x)\omega|\le K\Big\} \subset A$$

such that

$$\{(\Sigma\Sigma^*)(x,y,z,v), \beta(x,y,z,v) | (z,v)\in A\}$$

is convex.

Proof. See Buckdahn et al. [2010], Proposition 4.

□

The following Theorem will provide a subsequence of (5.31) (resp. (5.32)) such that the corresponding solutions of this subsequence converge.

Theorem 5.3.3. *Suppose Assumption 5.1.1 holds. Let* $(t,x) \in [0,T]\times\mathbb{R}^d$ *and* $(\delta_n)_{n\in\mathbb{N}} \subset (0,1]$ *with* $\lim_{n\to\infty}\delta_n = 0$. *Then there exists a reference Brownian stochastic basis* $\hat{\nu} = (\hat{\Omega}, \hat{\mathcal{F}}, \left(\hat{\mathcal{F}}_s\right)_{s\geq 0}, \hat{P}, \hat{W})$, *a triple* $(\bar{X}^{t,x}, \bar{Y}^{t,x}, \bar{Z}^{t,x}) \in \mathcal{S}^2_{\hat{\nu}}([t,T];\mathbb{R}^d) \times \mathcal{S}^2_{\hat{\nu}}([t,T];\mathbb{R}) \times \mathcal{H}^2_{\hat{\nu}}([t,T];\mathbb{R}^d)$ *and an admissible control* $\bar{u}^{t,x} \in \mathcal{U}_{\hat{\nu}}[t,T]$, *such that the following properties are fulfilled.*

1. *Let* $(X^{t,x,n}, Y^{t,x,n})_{n\in\mathbb{N}}$ *be the solution to system (5.31). Then there is a subsequence of* $(X^{t,x,n}, Y^{t,x,n})_{n\in\mathbb{N}}$ *that converges in distribution to* $(\bar{X}^{t,x}, \bar{Y}^{t,x})$, *which is the solution of the following system.*

$$\begin{cases} d\bar{X}^{t,x}_s = b(\bar{X}^{t,x}_s, \bar{u}^{t,x}_s)ds + \sigma(\bar{X}^{t,x}_s)d\hat{W}_s, \\ d\bar{Y}^{t,x}_s = -f(\bar{X}^{t,x}_s, \bar{u}^{t,x}_s)ds + \bar{Z}^{t,x}_s d\hat{W}_s, \quad s\in[t,T], \\ \bar{X}^{t,x}_t = x, \quad \bar{Y}^{t,x}_t = V(t,x). \end{cases} \tag{5.33}$$

2. *If a subsequence of* $(X^{t,x,n}, Y^{t,x,n})_{n\in\mathbb{N}}$ *converges in distribution, the same holds true for a subsequence of* $(X^{t,x,\delta_n}, Y^{t,x,\delta_n})_{n\in\mathbb{N}}$, *where* $(X^{t,x,\delta_n}, Y^{t,x,\delta_n})_{n\in\mathbb{N}}$ *denotes the solution to (5.30). Furthermore, the limits have identical law.*

3. *For all* $(t,x)\in[0,T]\times\mathbb{R}^d$, *we have*

$$\bar{Y}^{t,x}_t = V(t,x) = \inf_{u\in\mathcal{U}_{\hat{\nu}}[t,T]} J(t,x,u),$$

i.e., the admissible control $\bar{u}^{t,x} \in \mathcal{U}_{\hat{\nu}}[t,T]$ *is optimal for (5.33).*

Proof. 1.) The proof of property 1. is closely related to the proof of Theorem 2.5.3 in Yong and Zhou [1999] and goes back to Kushner [1975]. The proof will be carried out in several steps.
Consider equation (5.32) and define for any $(t,x)\in[0,T]\times\mathbb{R}^d$ and $n\in\mathbb{N}$

$$(\Phi^{t,x,n}_s)_{s\in[t,T]} = (\chi^{t,x,n}_s, B^{t,x,n}_s, S^{t,x,n}_s, W_s)_{s\in[t,T]},$$

where for $s\in[t,T]$

$$\begin{aligned} B^{t,x,n}_s &\equiv \int_t^s \beta(\chi^{t,x,n}_\tau, r^{t,x,n}_\tau)d\tau, \\ S^{t,x,n}_s &\equiv \int_t^s \Sigma(\chi^{t,x,n}_\tau, r^{t,x,n}_\tau)dW_\tau. \end{aligned}$$

In what follows we will consider some arbitrarily fixed $(t,x)\in[0,T]\times\mathbb{R}^d$. Without loss of generality, we can assume that $t=0$ for notational simplicity. With this fixed initial values we will omit the superscripts (t,x) for all involved processes. Therefore we write in the following for $n\in\mathbb{N}$

$$(\Phi^n_s)_{s\in[0,T]} = (\chi^n_s, B^n_s, S^n_s, W_s)_{s\in[0,T]},$$

where for $s \in [0, T]$

$$\begin{aligned} B_s^n &\equiv \int_0^s \beta(\chi_\tau^n, r_\tau^n) d\tau, \\ S_s^n &\equiv \int_0^s \Sigma(\chi_\tau^n, r_\tau^n) dW_\tau. \end{aligned}$$

Step 1: Show that the laws of $(\Phi_\cdot^n)_{n\in\mathbb{N}}$ ***are tight.***

In order to show that the laws of $(\Phi_\cdot^n)_{n\in\mathbb{N}}$ are tight we use the the following lemma (compare Theorem 2.1.10).

Lemma 5.3.4. *Let Assumption 5.1.1 hold. Then there exists a constant* $K > 0$ *(which depends on the arbitrarily fixed* $x \in \mathbb{R}^d$*), such that*

$$E[|\Phi_{s_1}^n - \Phi_{s_2}^n|^4] \leq K|s_1 - s_2|^2, \quad s_1, s_2 \in [0, T], n \in \mathbb{N}.$$

Proof. Let us fix $n \in \mathbb{N}$ and $0 \leq s_1 \leq s_2 \leq T$. Then from Theorem 2.4.4 we get

$$E\left[|\chi_{s_1}^n - \chi_{s_2}^n|^4\right] \leq K|s_1 - s_2|^2,$$

where K is a constant depending on the starting value x.
Moreover

$$\begin{aligned} E\left[|B_{s_1}^n - B_{s_2}^n|^4\right] &= E\left[\left|\int_{s_1}^{s_2} \beta(\chi_\tau^n, r_\tau^n) d\tau\right|^4\right] \\ &\leq |s_1 - s_2|^2 E\left[\left(\int_{s_1}^{s_2} |\beta(\chi_\tau^n, r_\tau^n)|^2 d\tau\right)^2\right] \\ &\leq |s_1 - s_2|^2 \int_{s_1}^{s_2} K_{grow}^2 E\left[(1 + |X_\tau|)^2\right] d\tau \\ &\leq K \cdot |s_1 - s_2|^2. \end{aligned}$$

Note that in the third line we used the linear growth property of function β (see Remark 5.3.1). In the last line, we used the fact that X has Lipschitz continuous coefficients with bounded growth such that its expected value can be estimated by a constant depending on the starting value x (see again Theorem 2.4.4).
We next estimate

$$\begin{aligned} E\left[|S_{s_1}^n - S_{s_2}^n|^4\right] &= E\left[\left|\int_{s_1}^{s_2} \Sigma(\chi_\tau^n, r_\tau^n) dW_\tau^n\right|^4\right] \\ &\leq E\left[\sup_{s_1 \leq s \leq s_2} \left|\int_{s_1}^{s} \Sigma(\chi_\tau^n, r_\tau^n) dW_\tau^n\right|^4\right] \\ &\leq \gamma E\left[\int_{s_1}^{s_2} \|\Sigma(\chi_\tau^n, r_\tau^n)\|^2 d\tau\right]^2 \end{aligned}$$

$$\leq \quad K \cdot |s_1 - s_2|^2.$$

Note that in the third line γ denotes a constant which enters because we used the inequality of Burkholder-Davis-Gundy. In the last step we used the fact that Σ is bounded (compare Remark 5.3.1).
Finally, since $W_{s_2-s_1}$ is Gaussian with mean zero and variance $I_d(s_2 - s_1)$, we obtain

$$E\left[|W_{s_1} - W_{s_2}|^4\right] = E\left[|W_{s_2-s_1}|^4\right] \leq K \cdot |s_1 - s_2|^2.$$

Combining the individual estimates yields

$$E\left[|\Phi^n_{s_1} - \Phi^n_{s_2}|^4\right] \leq K \cdot |s_1 - s_2|^2, \quad s_1, s_2 \in [t, T], n \in \mathbb{N},$$

for some constant K.

$\square$

Step 2: We introduce relaxed controls.

Let $n \in \mathbb{N}$ and $s \in [0, T]$. Recall that $\omega^n_s = \nabla_x V^{\delta_n}(s, X^{\delta_n}_s)$. Since $\nabla_x V^{\delta_n}$ is uniformly bounded in δ_n (see Proposition 5.2.4), and since σ is bounded as well, we can interpret $r^n_s = (\omega^n_s \sigma(X^n_s), u^{\delta_n}_s)$, $s \in [t, T]$, as a control with values in the compact set A of Proposition 5.3.2.
When introducing relaxed controls in Chapter 3, we showed in Remark 3.1.8 that it is possible to embed strict controls into the class of relaxed controls. Therefore, we can embed the controls r^n into the set of relaxed controls $\Lambda(A)$ by setting

$$\lambda_n(ds, da) = \lambda'_n(s, da)ds \equiv \delta_{r^n_s}(da)ds.$$

In this context, δ_z denotes the Dirac measure at $z \in A$.
Note that by Corollary 2.1.4 it follows that the set of relaxed controls $(\lambda_n)_{n\in\mathbb{N}}$ is tight, since A is compact.
Using the notation of relaxed controls, we can write the state process (5.32) as

$$\begin{aligned} d\chi^n_s &= \int_A \beta(\chi^n_s, a)\lambda'_n(s, da)ds + \int_A \Sigma(\chi^n_s, a)\lambda'_n(s, da)dW_s \\ &\equiv \tilde{\beta}(\chi^n_s, \lambda_n)ds + \tilde{\Sigma}(\chi^n_s, \lambda_n)dW_s, \quad s \in [0, T], \\ \chi^n_t &= \begin{pmatrix} x \\ V(t, x) \end{pmatrix}. \end{aligned} \tag{5.34}$$

Note that in contrast to (5.32) which describes a system of equations depending on all $(t, x) \in [0, T] \times \mathbb{R}^d$, we consider here the arbitrarily fixed initial values $x \in \mathbb{R}^d$ and $t = 0$ such that we omit the superscripts t and x. Here the functions $\tilde{\beta}$ and $\tilde{\Sigma}$ are defined similarly as in Definition 3.1.7, i.e.,

$$\tilde{\beta}(x_s, \lambda) \equiv \int_A \beta(x_s, a)\lambda'(s, da), \quad (s, x_s, \lambda) \in [0, T] \times \mathbb{R}^{d+1} \times \Lambda(A) \tag{5.35}$$

and

$$\tilde{\Sigma}(x_s, \lambda) \equiv \int_A \Sigma(x_s, a)\lambda'(s, da), \quad (s, x_s, \lambda) \in [0, T] \times \mathbb{R}^{d+1} \times \Lambda(A). \tag{5.36}$$

The tightness of $(\Phi^n_\cdot)_{n\in\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}}$ together imply the tightness of $(\Phi^n_\cdot, \lambda_n)_{n\in\mathbb{N}}$ as a sequence of $C\left([0, T]; \mathbb{R}^{4d}\right) \times \Lambda(A)$-valued random variables.

Step 3: Apply Skorohod's Theorem to $(\Phi^n, \lambda_n)_{n\in\mathbb{N}}$.

By Skorohod's Theorem (compare Theorem 2.1.12 and Corollary 2.1.13), one can choose a subsequence (still labeled by n) and get the existence of processes

$$\begin{cases} \left(\bar{\Phi}^n, \bar{\lambda}_n\right)_{n\in\mathbb{N}} := \left(\bar{\chi}^n, \bar{B}^n, \bar{S}^n, \bar{W}, \bar{\lambda}_n\right)_{n\in\mathbb{N}}, \\ \left(\bar{\Phi}, \bar{\lambda}\right) := \left(\bar{\chi}, \bar{B}, \bar{S}, \bar{W}, \bar{\lambda}\right), \end{cases}$$

on a suitable common probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ such that

$$\left(\bar{\Phi}^n_\cdot, \bar{\lambda}_n\right) \overset{\mathcal{L}}{\approx} \left(\Phi^n_\cdot, \delta_{r^n_\cdot}\right), \quad n \in \mathbb{N}, \tag{5.37}$$

and

$$\bar{\Phi}^n_s \text{ converges to } \bar{\Phi}_s \ \bar{P}\text{-a.s. uniformly for } s \in [0, T], \tag{5.38}$$

and

$$\bar{\lambda}_n \to \bar{\lambda} \text{ weakly in } \Lambda(A). \tag{5.39}$$

As in Chapter 3, Section 3.1.2, we introduce for any $f \in C([0, T] \times A)$ and $r \in [0, T]$ the function $f^r \in C([0, T] \times A)$ which takes the form

$$f^r(s, a) \equiv f(s \wedge r, a),$$

and we identify any $\lambda \in \Lambda(A)$ with a linear functional on $C([0, T] \times A)$ in writing

$$\lambda(f) \equiv \int_0^T \int_A f(s, a)\lambda(ds, da), \quad f \in C([0, T] \times A).$$

Let $(f_j)_{j\geq 1}$ be a countable dense subset (with respect to the supremum norm) of $C([0, T] \times A)$. Then for any $r \in [0, T]$, $\left(f^r_j\right)_{j\geq 1}$ is dense as well in the set $\{f^r | f \in C([0, T] \times A)\}$.

As shown in Section 3.1.2, a suitable filtration on $\Lambda(A)$ can be generated by

$$\mathcal{B}_s(\Lambda) = \sigma\left(\left\{\lambda \in \Lambda(A) | \lambda(f_j^r) \in (a,b)\right\} : r \le s \in \mathbb{Q}, j \ge 1, \text{ and } a,b \in \mathbb{Q}\right), \quad s \ge 0. \tag{5.40}$$

Now set for $n \in \mathbb{N}$ and $s \ge 0$

$$\begin{cases} \bar{\mathcal{F}}_{ns} := \left(\sigma\left(\bar{W}_r, \bar{\chi}_r^n : r \le s\right) \vee (\bar{\lambda}_n)^{-1}(\mathcal{B}_s(\Lambda))\right), \\ \bar{\mathcal{F}}_s := \left(\sigma\left(\bar{W}_r, \bar{\chi}_r : r \le s\right) \vee (\bar{\lambda})^{-1}(\mathcal{B}_s(\Lambda))\right). \end{cases}$$

Step 4: We show that $\bar{W}$ is an $\left(\bar{\mathcal{F}}_{ns}\right)_{s\ge 0}$-Brownian motion.

By the definition of $\mathcal{B}_s(\Lambda)$ and the fact that the σ-algebra generated by the cylinder sets of $C([0,T],\mathbb{R}^d)$ coincides with $\mathcal{B}(C([0,T];\mathbb{R}^d))$ (see Lemma 2.2.15) it follows that $\bar{\mathcal{F}}_{ns}$ with $n \in \mathbb{N}$ and $s \ge 0$ is the σ-field generated by $\bar{W}_{t_1}, \dots, \bar{W}_{t_\ell}$, $\bar{\chi}_{t_1}^n, \cdots \bar{\chi}_{t_\ell}^n$, $\bar{\lambda}_n(f_j^{t_1}), \cdots, \bar{\lambda}_n(f_j^{t_\ell})$, $0 \le t_1 \le t_2 \le \cdots \le t_\ell \le s$ and $j, \ell = 1, 2, \cdots$. Similar results hold for $\bar{\mathcal{F}}_s$.
In order to show that $\bar{W}$ is an $\left(\bar{\mathcal{F}}_{ns}\right)_{s\ge 0}$-Brownian motion, note first that obviously W is a $\sigma\left(W_r, \chi_r^n; r \le s\right) \vee (\lambda_n)^{-1}(\mathcal{B}_s(\Lambda))$-Brownian motion. Thus, for $\zeta, \ell, n \in \mathbb{N}$ we know by Proposition 2.2.12 that for any $0 \le r \le s \le T$ and any bounded continuous function g on $\mathbb{R}^{(2d+\zeta)\ell}$

$$E[g(\Psi^n)(W_s - W_r)] = 0,$$

where

$$\Psi^n \equiv \left\{W_{t_i}, \chi_{t_i}^n, \lambda_n(f_{j\alpha}^{t_i})\right\}, \quad 0 \le t_1 \le t_2 \le \cdots \le t_\ell \le s, \quad \alpha = 1, 2, ..., \zeta.$$

By (5.37) we have

$$\bar{E}[g(\bar{\Psi}^n)(\bar{W}_s - \bar{W}_r)] = 0,$$

where $\bar{E}[\cdots]$ denotes the expectation with respect to the probability measure $\bar{P}$ and

$$\bar{\Psi}^n \equiv \left\{\bar{W}_{t_i}, \bar{\chi}_{t_i}^n, \bar{\lambda}_n(f_{j\alpha}^{t_i})\right\}, \quad 0 \le t_1 \le t_2 \le \cdots \le t_\ell \le s, \quad \alpha = 1, 2, ..., \zeta. \tag{5.41}$$

For $\zeta, \ell, n \in \mathbb{N}$ one can similarly show that for any $0 \le r \le s \le T$ and any bounded continuous function g on $\mathbb{R}^{(2d+\zeta)\ell}$, we have

$$\bar{E}[g(\bar{\Psi}^n)(\bar{W}_s - \bar{W}_r)(\bar{W}_s - \bar{W}_r)^*] = (s-r)I_d.$$

Consequently, from Proposition 2.2.12, it follows that $\bar{W}$ is an $\left(\bar{\mathcal{F}}_{ns}\right)_{s\ge 0}$-Brownian motion (see also Remark 2.2.13).
Consider the SDE (5.34). Then by (5.37), for $n \in \mathbb{N}$ we have the following SDE on

$(\bar{\Omega}, \bar{\mathcal{F}}, \left(\bar{\mathcal{F}}_{ns}\right)_{s\geq 0}, \bar{P})$:

$$\begin{aligned}
\bar{\chi}_s^n &= \chi_0 + \int_0^s \tilde{\beta}(\bar{\chi}_\tau^n, \bar{\lambda}_n)d\tau + \int_0^s \tilde{\Sigma}(\bar{\chi}_\tau^n, \bar{\lambda}_n)d\bar{W}_\tau \qquad (5.42)\\
&= \chi_0 + \int_0^s \int_A \beta(\bar{\chi}_\tau^n, a)\bar{\lambda}_n'(\tau, da)d\tau + \int_0^s \int_A \Sigma(\bar{\chi}_\tau^n, a)\bar{\lambda}_n'(\tau, da)d\bar{W}_\tau\\
&=: \chi_0 + \bar{B}_s^n + \bar{S}_s^n, \quad s \in [0,T]. \qquad (5.43)
\end{aligned}$$

Note that the integrals are well-defined due to the fact that $\bar{W}$ is an $\left(\bar{\mathcal{F}}_{ns}\right)_{s\geq 0}$-Brownian motion.
Letting $n \to \infty$ in (5.43) and noting (5.38), we get

$$\bar{\chi}_s = \chi_0 + \bar{B}_s + \bar{S}_s, \quad s \in [0,T], \quad \bar{P}\text{-a.s.} \qquad (5.44)$$

Remark 5.3.5. For $n \in \mathbb{N}$, by definition $\bar{S}^n$ is a continuous $\left(\bar{\mathcal{F}}_{ns}\right)_{s\geq 0}$-martingale. Furthermore, from (5.42), it follows that

$$\left[\bar{S}^n\right]_s = \int_0^s \tilde{\Sigma}\tilde{\Sigma}^*(\bar{\chi}_\tau^n, \bar{\lambda}_n)d\tau, \quad s \in [0,T],$$

where $\left[\bar{S}^n\right]$ is the quadratic variation of $\bar{S}^n$ (compare Definition 2.2.6). Hence $\bar{S}^n\bar{S}^{n*} - \int_0^\cdot \tilde{\Sigma}\tilde{\Sigma}^*(\bar{\chi}_\tau^n, \bar{\lambda}_n)d\tau$ is an $\left(\bar{\mathcal{F}}_{ns}\right)_{s\geq 0}$-martingale.

Step 5: We show that $\bar{S}$ is an $\left(\bar{\mathcal{F}}_s\right)_{s\geq 0}$-martingale.

Again, for $\zeta, \ell, n \in \mathbb{N}$, $0 \leq s \leq T$, define $\bar{\Psi}^n$ as in (5.41) and let

$$\begin{aligned}
\bar{\Psi} \quad &\equiv \quad \left\{\bar{W}_{t_i}, \bar{\chi}_{t_i}, \bar{\lambda}(f_{j\alpha}^{t_i})\right\}, \qquad (5.45)\\
0 \quad &\leq t_1 \leq t_2 \leq \cdots \leq t_\ell \leq s, \quad j, \ell = 1, 2, \cdots, \quad \alpha = 1, 2, ..., \zeta.
\end{aligned}$$

Let g denote an arbitrary bounded continuous function on $\mathbb{R}^{(2d+\zeta)\ell}$. Then by (5.38) and (5.39) for any $0 \leq r \leq s \leq T$

$$g(\bar{\Psi}^n)(\bar{S}_s^n - \bar{S}_r^n) \overset{n\to\infty}{\longrightarrow} g(\bar{\Psi})(\bar{S}_s - \bar{S}_r), \quad \bar{P} - a.s.$$

In addition, by the Burkholder-Davis-Gundy inequality, we have for any $0 \leq r \leq s \leq T$

$$\begin{aligned}
\sup_{n\in\mathbb{N}} \bar{E}\left[\left|g(\bar{\Psi}^n)(\bar{S}_s^n - \bar{S}_r^n)\right|\right] &= \sup_{n\in\mathbb{N}} \bar{E}\left[\left|g(\bar{\Psi}^n)\int_r^s \tilde{\Sigma}(\bar{\chi}_\tau^n, \bar{\lambda}_n)d\bar{W}_\tau\right|\right]\\
&\leq C \sup_{n\in\mathbb{N}} \sup_{s\in[0,T]} \bar{E}\left[\left|\int_0^s \tilde{\Sigma}(\bar{\chi}_\tau^n, \bar{\lambda}_n)d\bar{W}_\tau\right|\right]\\
&\leq C \sup_{n\in\mathbb{N}} \bar{E}\left[\int_0^T \left|\tilde{\Sigma}(\bar{\chi}_\tau^n, \bar{\lambda}_n)\right|^2 d\tau\right]^{\frac{1}{2}}
\end{aligned}$$

$$\leq \quad C,$$

for a constant C that may vary from line to line.
Consequently, applying the dominated convergence theorem yields

$$0 = \bar{E}\left[g(\bar{\Psi}^n)(\bar{S}_s^n - \bar{S}_r^n)\right] \to \bar{E}\left[g(\bar{\Psi})(\bar{S}_s - \bar{S}_r)\right], \quad 0 \leq r \leq s \leq T.$$

This proves that $\bar{S}$ is a continuous $\left(\bar{\mathcal{F}}_s\right)_{s\geq 0}$-martingale in view of Proposition 2.2.12.

Step 6: We characterize the limit behavior of the coefficients of (5.42).

Next, we consider the coefficients of (5.42) and define the following sequences for $s \in [0, T]$:

$$\alpha^n(s) = \tilde{\Sigma}\tilde{\Sigma}^*(\bar{\chi}_s^n, \bar{\lambda}_n), \quad n \in \mathbb{N},$$

and for $i = 1, ..., d$

$$\eta_i^n(s) = \tilde{\beta}_i(\bar{\chi}_s^n, \bar{\lambda}_n), \quad n \in \mathbb{N}, \tag{5.46}$$

where $\tilde{\beta}_i$ denotes the i-th element of function $\tilde{\beta}$. The aim is to assess the limit behavior of these sequences, starting with the first one.
By Assumption 5.1.1, $\sup_{n\in\mathbb{N}} \bar{E} \int_0^T |\alpha^n(s)|^2 ds < \infty$, and hence $(\alpha^n)_{n\in\mathbb{N}}$ is weakly relatively compact in the space $L^2([0, T] \times \bar{\Omega}; \mathbb{S}^{d+1})$. We can then find a subsequence (still labeled by n) and a function $\alpha \in L^2([0, T] \times \bar{\Omega}, \mathbb{S}^{d+1})$ such that

$$\alpha^n \to \alpha \text{ weakly in } L^2([0, T] \times \bar{\Omega}, \mathbb{S}^{d+1}). \tag{5.47}$$

Recall that weak convergence in this context was defined in Definition 2.1.14.
We obtain the following additional property.

Lemma 5.3.6. *For $1 \leq i, j \leq d$ and almost all $(s, \omega) \in [0, T] \times \bar{\Omega}$*

$$\varliminf_{n\to\infty} \alpha_{ij}^n(s, \omega) \leq \alpha_{ij}(s, \omega) \leq \varlimsup_{n\to\infty} \alpha_{ij}^n(s, \omega), \tag{5.48}$$

where for any $n \in \mathbb{N}$ α_{ij}^n denotes the ij-th entry of α^n.

Proof. If (5.48) is not true and for (s, ω) on a set $S \subseteq [0, T] \times \bar{\Omega}$ of positive measure,

$$\varliminf_{n\to\infty} \alpha_{ij}^n(s, \omega) > \alpha_{ij}(s, \omega),$$

then we have by Fatou's lemma that

$$\varliminf_{n\to\infty} \int_S \alpha_{ij}^n(s, \omega) ds d\bar{P}(\omega) > \int_S \alpha_{ij}(s, \omega) ds d\bar{P}(\omega),$$

which is a contradiction to (5.47). The same can be said for the $\varlimsup$. This proves (5.48).

□

By the Lipschitz continuity of the coefficients and by (5.38) we have for almost all (s, ω),

$$\begin{cases} \underline{\lim}_{n\to\infty} \alpha^n(s,\omega) = \underline{\lim}_{n\to\infty} \tilde{\Sigma}\tilde{\Sigma}^*(\bar{\chi}_s(\omega), \bar{\lambda}_n), \\ \overline{\lim}_{n\to\infty} \alpha^n(s,\omega) = \overline{\lim}_{n\to\infty} \tilde{\Sigma}\tilde{\Sigma}^*(\bar{\chi}_s(\omega), \bar{\lambda}_n). \end{cases} \tag{5.49}$$

Let $l(\chi, a)$ denote the set $\{(\Sigma\Sigma^*)(\chi, a), \beta(\chi, a)\}$ for all $(\chi, a) \in \mathbb{R}^{d+1} \times A$, where A stands for the compact set of Proposition 5.3.2, and let $\tilde{l}(\chi, \lambda)$ denote the set $\left\{(\tilde{\Sigma}\tilde{\Sigma}^*)(\chi, \lambda), \tilde{\beta}(\chi, \lambda)\right\}$ for all $(\chi, \lambda) \in \mathbb{R}^{d+1} \times \Lambda(A)$. As in equations (5.35) and (5.36) the two sets are connected by

$$\tilde{l}(\chi_s, \lambda) = \int_A l(\chi_s, a)\lambda'(s, da), \quad (s, \chi_s, \lambda) \in [0,T] \times \mathbb{R}^{d+1} \times \Lambda(A).$$

From Proposition 5.3.2 we know that for each $\chi \in \mathbb{R}^{d+1}$, $l(\chi, A)$ is a convex set of $\mathbb{R}^{(d+1)^2+(d+1)}$. It follows that for each $(s, \chi_s) \in [0,T] \times \mathbb{R}^{d+1}$ and each probability measure λ on A, $\tilde{l}(\chi_s, \lambda) = \int_A l(\chi_s, a)\lambda'(s, da)$ belongs to the closed convex set $l(\chi, A)$. Therefore, combining (5.48) and (5.49) gives

$$\alpha_{ij}(s,\omega) \in (\Sigma\Sigma^*)_{ij}(\bar{\chi}_s(\omega), A), \quad \text{for a.e. } (s,\omega) \in [0,T] \times \bar{\Omega}, \quad i, j = 1, ..., d. \tag{5.50}$$

Similarly, for η_i^n, $i = 1, ..., d$, $n \in \mathbb{N}$, defined in (5.46) one can prove that there are $\eta_i \in L^2([0,T] \times \bar{\Omega}; \mathbb{R})$ such that

$$\eta_i^n \to \eta_i \quad (i = 1, ..., d) \text{ weakly in } L^2([0,T] \times \bar{\Omega}, \mathbb{R}), \tag{5.51}$$

and

$$\eta_i(s,\omega) \in \beta_i(\bar{\chi}_s(\omega), A), \quad \text{for a.e. } (s,\omega) \in [0,T] \times \bar{\Omega}, \quad i = 1, ..., d. \tag{5.52}$$

By (5.50), (5.52), Remark 5.1.3 and the measurable selection theorem (see Li and Yong [1991], p. 102, Corollary 2.26) there is an A-valued, $\left(\bar{\mathcal{F}}_s\right)_{s\geq 0}$-adapted process $\bar{r}$ such that

$$(\eta, \alpha)(s,\omega) = (\beta, \Sigma\Sigma^*)(\bar{\chi}_s(\omega), \bar{r}_s), \quad (s,\omega) \in [0,T] \times \bar{\Omega}. \tag{5.53}$$

Consequently, by introducing relaxed controls, we finally obtain in the limit the existence of a strict control.

Summarizing (5.47), (5.51) and (5.53) it follows that for $s \in [0,T]$

$$\alpha^n(s,\omega) \to \alpha(s,\omega) = \Sigma\Sigma^*(\bar{\chi}_s(\omega), \bar{r}_s) \text{ weakly in } L^2([0,T] \times \bar{\Omega}, \mathbb{R}^{d+1}),$$

and

$$\eta_i^n(s,\omega) \to \eta_i(s,\omega) = \beta_i(\bar{\chi}_s(\omega), \bar{r}_s) \text{ weakly in } L^2([0,T] \times \bar{\Omega}, \mathbb{R}).$$

Consequently, we have for any $s_1, s_2 \in [0,T]$,

$$\int_{s_1}^{s_2} \alpha^n(\tau,\omega)d\tau \to \int_{s_1}^{s_2} \Sigma\Sigma^*(\bar{\chi}_\tau(\omega), \bar{r}_\tau)d\tau, \text{ weakly in } L^2(\bar{\Omega}), \tag{5.54}$$

and

$$\int_{s_1}^{s_2} \eta^n(\tau,\omega)d\tau \to \int_{s_1}^{s_2} \beta(\bar{\chi}_\tau(\omega), \bar{r}_\tau)d\tau, \text{ weakly in } L^2(\bar{\Omega}). \tag{5.55}$$

Step 7: We show the form of $\bar{\chi}$ as a Stochastic Differential Equation.

Combining the preceding definitions and results, we obtain for any $s \in [0,T]$

$$\bar{B}^n_s \underset{5.43}{=} \int_0^s \tilde{\beta}(\bar{\chi}^n_\tau, \bar{\lambda}_n)d\tau \underset{5.46}{=} \int_0^s \eta^n(\tau)d\tau \underset{5.55}{\to} \int_0^s \beta(\bar{\chi}_\tau, \bar{r}_\tau)d\tau, \quad s \in [0,T]. \tag{5.56}$$

Since by (5.44) $\bar{B}^n_s \to \bar{B}_s$ for all $s \in [0,T]$, we get together with (5.56) that

$$\bar{B}_s = \int_0^s \beta(\bar{\chi}_\tau, \bar{r}_\tau)d\tau. \tag{5.57}$$

After we just specified the form of $\bar{B}_s$, we now turn to the process $\bar{S}_s$ in equation (5.44). By the dominated convergence theorem we get for any bounded continuous function g on $\mathbb{R}^{(2d+\zeta)\ell}$, $\zeta, \ell \in \mathbb{N}$,

$$\lim_{n\to\infty} \bar{E}\left[|g(\bar{\Psi}^n)|^2\right]^{\frac{1}{2}} = \bar{E}\left[|g(\bar{\Psi})|^2\right]^{\frac{1}{2}},$$

or equivalently

$$\lim_{n\to\infty} \left\|g(\bar{\Psi}^n)\right\|_{L^2} = \left\|g(\bar{\Psi})\right\|_{L^2},$$

where $\bar{\Psi}^n$, resp. $\bar{\Psi}$ are defined as in (5.41) resp. (5.45).
In addition, from (5.38) and (5.39) it follows that $g(\bar{\Psi}^n) \to g(\bar{\Psi})$ $\bar{P}$-a.s. as $n \to \infty$. Since pointwise convergence and the convergence of the L^p-norms imply strong convergence, we obtain

$$g(\bar{\Psi}^n) \to g(\bar{\Psi}), \quad \text{strongly in } L^2(\bar{\Omega}). \tag{5.58}$$

Together with (5.54), we obtain

$$\bar{E}\left(g(\bar{\Psi}^n)\int_{s_1}^{s_2} \alpha^n(\tau)d\tau\right) \to \bar{E}\left(g(\bar{\Psi})\int_{s_1}^{s_2} \Sigma\Sigma^*(\bar{\chi}_\tau, \bar{r}_\tau)d\tau\right).$$

Since $\bar{S}^n\bar{S}^{n*} - \int_0^\cdot \alpha^n(\tau)d\tau$ is an $\left(\bar{\mathcal{F}}_{ns}\right)_{s\geq 0}$-martingale (see Remark 5.3.5) for $s \in [0,T]$ and since $\bar{S}^n \to \bar{S}$ $\bar{P}$-a.s. (compare equations (5.43) and (5.44)), it follows by Proposition 2.2.12 that for $s \in [0,T]$

$$0 = \bar{E}\left[g(\bar{\Psi}^n)\left(\bar{S}^n_s\bar{S}^{n*}_s - \int_0^s \alpha^n(\tau)d\tau\right)\right] \to \bar{E}\left[g(\bar{\Psi})\left(\bar{S}_s\bar{S}^*_s - \int_0^s \Sigma\Sigma^*(\bar{\chi}_\tau, \bar{r}_\tau)d\tau\right)\right].$$

Consequently, $\bar{S}\bar{S}^* - \int_0^\cdot \Sigma\Sigma^*(\bar{\chi}_\tau, \bar{r}_\tau)d\tau$, $s \in [0,T]$, is an $\left(\bar{\mathcal{F}}_s\right)_{s\geq 0}$-martingale. This implies

$$\left[\bar{S}\right]_s = \int_0^s \Sigma\Sigma^*(\bar{\chi}_\tau, \bar{r}_\tau)d\tau, \quad s \in [0,T].$$

Since we already showed in Step 5 that $\bar{S}$ is a continuous $\left(\bar{\mathcal{F}}_s\right)_{s\geq 0}$-martingale, we can apply the martingale representation theorem (see Theorem 2.3.3) and it follows that there is an extension $(\hat{\Omega}, \hat{\mathcal{F}}, \left(\hat{\mathcal{F}}_s\right)_{s\geq 0}, \hat{P})$ of $(\bar{\Omega}, \bar{\mathcal{F}}, \left(\bar{\mathcal{F}}_s\right)_{s\geq 0}, \bar{P})$ on which a d-dimensional $\left(\hat{\mathcal{F}}_s\right)_{s\geq 0}$-Brownian motion $\hat{W}$ lives such that

$$\bar{S}_s = \int_0^s \Sigma(\bar{\chi}_\tau, \bar{r}_\tau)d\hat{W}_\tau, \quad s \in [0,T]. \tag{5.59}$$

Finally, combining (5.44), (5.57) and (5.59), we obtain that $\bar{\chi}$ can be written in form of the following stochastic differential equation.

$$\bar{\chi}_s = \chi_0 + \int_0^s \beta(\bar{\chi}_\tau, \bar{r}_\tau)d\tau + \int_0^s \Sigma(\bar{\chi}_\tau, \bar{r}_\tau)d\hat{W}_r, \quad s \in [0,T]. \tag{5.60}$$

Replacing β and Σ by their definition and setting $\bar{\chi} = \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix}$ and $\bar{r} = (\bar{Z}, \bar{u})$, this system is seen to be

$$\begin{cases} d\bar{X}_s = b(\bar{X}_s, \bar{u}_s)ds + \sigma(\bar{X}_s)d\hat{W}_s, \\ d\bar{Y}_s = -f(\bar{X}_s, \bar{u}_s)ds + \bar{Z}_s d\hat{W}_s, \quad s \in [0,T], \\ \bar{X}_t = x, \quad \bar{Y}_t = V(t,x). \end{cases} \tag{5.61}$$

Since we considered in Step 1 to 7 some arbitrarily fixed $(t,x) \in [0,T] \times \mathbb{R}^d$, we can generalize the statement of the proof of 1.) for every $(t,x) \in [0,T] \times \mathbb{R}^d$. For that it is necessary to reintroduce the superscripts t and x in order to show the dependence of the involved processes on these initial parameters. This means for every $(t,x) \in [0,T] \times \mathbb{R}^d$ there exists a reference Brownian stochastic basis $\hat{\nu} = (\hat{\Omega}, \hat{\mathcal{F}}, \left(\hat{\mathcal{F}}_s\right)_{s\geq 0}, \hat{P}, \hat{W})$, a triple $(\bar{X}^{t,x}, \bar{Y}^{t,x}, \bar{Z}^{t,x}) \in \mathcal{S}^2_{\hat{\nu}}([t,T];\mathbb{R}^d) \times \mathcal{S}^2_{\hat{\nu}}([t,T];\mathbb{R}) \times \mathcal{H}^2_{\hat{\nu}}([t,T];\mathbb{R}^d)$ and an admissible control $\bar{u}^{t,x} \in \mathcal{U}_{\hat{\nu}}[t,T]$, such that there is a subsequence of $(X^{t,x,n}, Y^{t,x,n})_{n\in\mathbb{N}}$ that converges in distribution to $(\bar{X}^{t,x}, \bar{Y}^{t,x})$. For $(t,x) \in [0,T] \times \mathbb{R}^d$ and $n \in \mathbb{N}$ $(X^{t,x,n}, Y^{t,x,n})_{n\in\mathbb{N}}$ is the solution to the system defined in (5.31) and $(\bar{X}^{t,x}, \bar{Y}^{t,x})$ is the solution of

$$\begin{cases} d\bar{X}^{t,x}_s = b(\bar{X}^{t,x}_s, \bar{u}^{t,x}_s)ds + \sigma(\bar{X}^{t,x}_s)d\hat{W}_s, \\ d\bar{Y}^{t,x}_s = -f(\bar{X}^{t,x}_s, \bar{u}^{t,x}_s)ds + \bar{Z}^{t,x}_s d\hat{W}_s, \quad s \in [t,T], \\ \bar{X}^{t,x}_t = x, \quad \bar{Y}^{t,x}_t = V(t,x). \end{cases} \tag{5.62}$$

Consequently, 1.) is shown.

2.) We now prove the second claim.
We know that for any $(t,x) \in [0,T] \times \mathbb{R}^d$ a subsequence of the sequence of processes $(X^{t,x,n}, Y^{t,x,n})_{n\in\mathbb{N}}$ converges in law. We will show next that the same holds true for $(X^{t,x,\delta_n}, Y^{t,x,\delta_n})_{n\in\mathbb{N}}$ and the limits have identical law. This will be done by proving for any $(t,x) \in [0,T] \times \mathbb{R}^d$

$$E\left[\sup_{s\in[t,T]} |X_s^{t,x,\delta_n} - X_s^{t,x,n}|^2\right] \leq K\delta_n^2, \tag{5.63}$$

and

$$E\left[\sup_{s\in[t,T]} |Y_s^{t,x,\delta_n} - Y_s^{t,x,n}|^2\right] \leq K\delta_n^2. \tag{5.64}$$

We start with proving inequality (5.63).
Recall that for $(t,x) \in [0,T] \times \mathbb{R}^d$ and $n \in \mathbb{N}$

$$\begin{aligned} dX_s^{t,x,\delta_n} &= b(X_s^{t,x,\delta_n}, u_s^{t,x,\delta_n})ds + \sigma^{\delta_n}(X_s^{t,x,\delta_n})dW_s, \quad X_t^{t,x,\delta_n} = x, \\ dX_s^{t,x,n} &= b(X_s^{t,x,n}, u_s^{t,x,\delta_n})ds + \sigma(X_s^{t,x,n})dW_s, \quad X_t^{t,x,n} = x, \quad s \in [t,T], \end{aligned}$$

where $(\delta_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers converging to 0. The following estimates are very similar to the estimates in the proof of claim 2) in Proposition 5.2.4.
We have for any $(t,x) \in [0,T] \times \mathbb{R}^d$ and $n \in \mathbb{N}$

$$\begin{aligned} & |X_s^{t,x,\delta_n} - X_s^{t,x,n}|^2 \\ \leq\ & 2\left(\sup_{s\in[t,T]} \left|\int_t^s (b(X_\tau^{t,x,\delta_n}, u_\tau^{t,x,\delta_n}) - b(X_\tau^{t,x,n}, u_\tau^{t,x,\delta_n}))d\tau\right|^2\right. \\ +\ & \left.\sup_{s\in[t,T]} \left|\int_t^s (\sigma^{\delta_n}(X_\tau^{t,x,\delta_n}) - \sigma(X_\tau^{t,x,n}))dW_\tau\right|^2\right). \end{aligned}$$

Let in the following K be some constant that may vary from line to line. Then it follows by using the Burkholder-Davis-Gundy inequality, the Lipschitz continuity of b and σ and inequality (5.6) that for any $(t,x) \in [0,T] \times \mathbb{R}^d$ and $n \in \mathbb{N}$

$$\begin{aligned} & E\left[\sup_{s\in[t,T]} |X_s^{t,x,\delta_n} - X_s^{t,x,n}|^2\right] \\ \leq\ & K\left(E\left[\sup_{s\in[t,T]} \left(\int_t^s \left|b(X_\tau^{t,x,\delta_n}, u_\tau^{t,x,\delta_n}) - b(X_\tau^{t,x,n}, u_\tau^{t,x,\delta_n})\right| d\tau\right)^2\right]\right. \\ +\ & \left. E\left[\int_t^T \left\|\sigma^{\delta_n}(X_\tau^{t,x,\delta_n}) - \sigma(X_\tau^{t,x,n})\right\|^2 d\tau\right]\right) \\ \leq\ & K\left(E\left[\int_t^T \left|b(X_\tau^{t,x,\delta_n}, u_\tau^{t,x,\delta_n}) - b(X_\tau^{t,x,n}, u_\tau^{t,x,\delta_n})\right|^2 d\tau\right]\right. \end{aligned}$$

$$
\begin{aligned}
&+ \quad E\left[\int_t^T \left\|\sigma^{\delta_n}(X_\tau^{t,x,\delta_n}) - \sigma(X_\tau^{t,x,\delta_n})\right\|^2 d\tau\right] + E\left[\int_t^T \left\|\sigma(X_\tau^{t,x,\delta_n}) - \sigma(X_\tau^{t,x,n})\right\|^2 d\tau\right]\Bigg) \\
&\leq \quad K\left(E\left[\int_t^T \left|X_\tau^{t,x,\delta_n} - X_\tau^{t,x,n}\right|^2 d\tau\right] + \delta_n^2\right).
\end{aligned}
$$

Applying Gronwall's lemma gives the existence of a constant still named K such that for all $(t,x) \in [0,T] \times \mathbb{R}^d$ and $n \in \mathbb{N}$

$$
E\left[\sup_{s\in[t,T]} |X_s^{t,x,\delta_n} - X_s^{t,x,n}|^2\right] \leq K\delta_n^2.
$$

Therefore (5.63) is shown.

Next, we will prove inequality (5.64).
Recall that for $(t,x) \in [0,T] \times \mathbb{R}^d$ and $n \in \mathbb{N}$

$$
\begin{aligned}
dY_s^{t,x,\delta_n} &= -f^{\delta_n}(X_s^{t,x,\delta_n}, u_s^{t,x,\delta_n}) + Z_s^{t,x,\delta_n} dW_s, \quad Y_t^{t,x,\delta_n} = V^{\delta_n}(t,x), \\
dY_s^{t,x,n} &= -f(X_s^{t,x,n}, u_s^{t,x,\delta_n})ds + w_s^n \sigma(X_s^{t,x,n}) dW_s, \quad Y_t^{t,x,n} = V(t,x), \quad s \in [t,T].
\end{aligned}
$$

By applying Itô's formula, and then estimating further $E\left[\sup_{s\in[t,T]} |Y_s^{t,x,\delta_n} - Y_s^{t,x,n}|^2\right]$ we obtain

$$
\begin{aligned}
&E\left[\sup_{s\in[t,T]} |Y_s^{t,x,\delta_n} - Y_s^{t,x,n}|^2\right] \\
\leq \quad & E\left[|V^{\delta_n}(t,x) - V(t,x)|^2\right] + E\left[\int_t^T |Z_\tau^{t,x,\delta_n} - w_\tau^n \sigma(X_s^{t,x,n})|^2 d\tau\right] \\
+ \quad & 2E\left[\sup_{s\in[t,T]} \left|\int_t^s \left\langle Y_\tau^{t,x,\delta_n} - Y_\tau^{t,x,n}, (Z_\tau^{t,x,\delta_n} - w_\tau^n \sigma(X_s^{t,x,n}))dW_\tau\right\rangle\right|\right] \\
+ \quad & 2E\left[\sup_{s\in[t,T]} \left|\int_t^s \left\langle Y_\tau^{t,x,\delta_n} - Y_\tau^{t,x,n}, f^{\delta_n}(X_\tau^{t,x,\delta_n}, u_\tau^{t,x,\delta_n}) - f(X_\tau^{t,x,n}, u_\tau^{t,x,\delta_n})\right\rangle d\tau\right|\right] \\
\leq \quad & E\left[|V^{\delta_n}(t,x) - V(t,x)|^2\right] + E\left[\int_t^T |Z_\tau^{t,x,\delta_n} - w_\tau^n \sigma(X_s^{t,x,n})|^2 d\tau\right] \\
+ \quad & 2E\left[\sup_{s\in[t,T]} \left|\int_t^s \left\langle Y_\tau^{t,x,\delta_n} - Y_\tau^{t,x,n}, (Z_\tau^{t,x,\delta_n} - w_\tau^n \sigma(X_s^{t,x,n}))dW_\tau\right\rangle\right|\right] \\
+ \quad & 2E\left[\sup_{s\in[t,T]} \left|\int_t^s \left\langle Y_\tau^{t,x,\delta_n} - Y_\tau^{t,x,n}, f^{\delta_n}(X_\tau^{t,x,\delta_n}, u_\tau^{t,x,\delta_n}) - f^{\delta_n}(X_\tau^{t,x,n}, u_\tau^{t,x,\delta_n})\right\rangle d\tau\right|\right] \\
+ \quad & 2E\left[\sup_{s\in[t,T]} \left|\int_t^s \left\langle Y_\tau^{t,x,\delta_n} - Y_\tau^{t,x,n}, f^{\delta_n}(X_\tau^{t,x,n}, u_\tau^{t,x,\delta_n}) - f(X_\tau^{t,x,n}, u_\tau^{t,x,\delta_n})\right\rangle d\tau\right|\right]
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & E\left[|V^{\delta_n}(t,x)-V(t,x)|^2\right]+E\left[\int_t^T |Z_\tau^{t,x,\delta_n}-w_\tau^n\sigma(X_s^{t,x,n})|^2 d\tau\right]\\
+\ & 2E\left[\sup_{s\in[t,T]}\left|\int_t^s \left\langle Y_\tau^{t,x,\delta_n}-Y_\tau^{t,x,n},(Z_\tau^{t,x,\delta_n}-w_\tau^n\sigma(X_s^{t,x,n}))dW_\tau\right\rangle\right|\right]\\
+\ & 2CE\left[\sup_{s\in[t,T]}\left|\int_t^s \left\langle Y_\tau^{t,x,\delta_n}-Y_\tau^{t,x,n},X_\tau^{t,x,\delta_n}-X_\tau^{t,x,n}\right\rangle d\tau\right|\right]\\
+\ & 2E\left[\sup_{s\in[t,T]}\left|\int_t^s \left\langle Y_\tau^{t,x,\delta_n}-Y_\tau^{t,x,n},f^{\delta_n}(X_\tau^{t,x,n},u_\tau^{t,x,\delta_n})-f(X_\tau^{t,x,n},u_\tau^{t,x,\delta_n})\right\rangle d\tau\right|\right],
\end{aligned}
$$

where C denotes the Lipschitz constant of f.
From the Burkholder-Davis-Gundy inequality, we know that there exists some constant γ such that for any $(t,x)\in[0,T]\times\mathbb{R}^d$ and $n\in\mathbb{N}$

$$
\begin{aligned}
& E\left[\sup_{s\in[t,T]}|Y_s^{t,x,\delta_n}-Y_s^{t,x,n}|^2\right] && (5.65)\\
\leq\ & E\left[|V^{\delta_n}(t,x)-V(t,x)|^2\right]+E\left[\int_t^T |Z_\tau^{t,x,\delta_n}-w_\tau^n\sigma(X_s^{t,x,n})|^2 d\tau\right]\\
+\ & 2\gamma E\left[\int_t^T |Y_\tau^{t,x,\delta_n}-Y_\tau^{t,x,n}|^2\cdot|Z_\tau^{t,x,\delta_n}-w_\tau^n\sigma(X_s^{t,x,n})|^2 d\tau\right]^{\frac{1}{2}} && (5.66)\\
+\ & 2CE\left[\sup_{s\in[t,T]}|Y_\tau^{t,x,\delta_n}-Y_\tau^{t,x,n}|\int_t^T |X_\tau^{t,x,\delta_n}-X_\tau^{t,x,n}|d\tau\right]\\
+\ & 2E\left[\sup_{s\in[t,T]}|Y_\tau^{t,x,\delta_n}-Y_\tau^{t,x,n}|\int_t^T |f^{\delta_n}-f|(X_\tau^{t,x,n},u_\tau^{t,x,\delta_n})d\tau\right].
\end{aligned}
$$

Then, we obtain by applying Young's inequality for any $\epsilon>0$, $(t,x)\in[0,T]\times\mathbb{R}^d$ and $n\in\mathbb{N}$

$$
\begin{aligned}
& E\left[\sup_{s\in[t,T]}|Y_s^{t,x,\delta_n}-Y_s^{t,x,n}|^2\right]\\
\leq\ & E\left[|V^{\delta_n}(t,x)-V(t,x)|^2\right]+E\left[\int_t^T |Z_\tau^{t,x,\delta_n}-w_\tau^n\sigma(X_s^{t,x,n})|^2 d\tau\right]\\
+\ & \frac{\gamma}{\epsilon}E\left[\sup_{s\in[t,T]}|Y_s^{t,x,\delta_n}-Y_s^{t,x,n}|^2\right]+\gamma\epsilon E\left[\int_t^T |Z_\tau^{t,x,\delta_n}-w_\tau^n\sigma(X_s^{t,x,n})|^2 d\tau\right]\\
+\ & \frac{C}{\epsilon}E\left[\sup_{s\in[t,T]}|Y_s^{t,x,\delta_n}-Y_s^{t,x,n}|^2\right]+C\epsilon E\left[\int_t^T |X_\tau^{t,x,\delta_n}-X_\tau^{t,x,n}|d\tau\right]^2\\
+\ & \frac{1}{\epsilon}E\left[\sup_{s\in[t,T]}|Y_s^{t,x,\delta_n}-Y_s^{t,x,n}|^2\right]+\epsilon E\left[\int_t^T |f^{\delta_n}-f|(X_\tau^{t,x,n},u_\tau^{t,x,\delta_n})d\tau\right]^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \; E\left[|V^{\delta_n}(t,x)-V(t,x)|^2\right]+(1+\gamma\epsilon)E\left[\int_t^T |Z_\tau^{t,x,\delta_n}-w_\tau^n\sigma(X_s^{t,x,n})|^2 d\tau\right] \\
&+ \; \frac{1+\gamma+C}{\epsilon}E\left[\sup_{s\in[t,T]}|Y_s^{t,x,\delta_n}-Y_s^{t,x,n}|^2\right]+C\epsilon E\left[\int_t^T |X_\tau^{t,x,\delta_n}-X_\tau^{t,x,n}|d\tau\right]^2 \\
&+ \; \epsilon(T-t)\delta_n^2.
\end{aligned}
$$

In the proof of Proposition 5.2.4, we show with the help of Itô's formula that the process Z^{t,x,δ_n} takes the following form for any $(t,x)\in[0,T]\times\mathbb{R}^d$ and $n\in\mathbb{N}$:

$$
Z_s^{\delta_n}=\nabla_x V^{\delta_n}(s,X_s^{t,x,\delta_n})\sigma^{\delta_n}(X_s^{t,x,\delta_n}),\quad s\in[t,T].
$$

In addition, recall that $w_s^n=\nabla_x V^{\delta_n}(s,X_s^{t,x,\delta_n})$ for any $(t,x)\in[0,T]\times\mathbb{R}^d$, $n\in\mathbb{N}$ and $s\in[t,T]$. Therefore, we may estimate for any $(t,x)\in[0,T]\times\mathbb{R}^d$ and $n\in\mathbb{N}$

$$
\begin{aligned}
& E\left[\int_t^T |Z_\tau^{t,x,\delta_n}-w_\tau^n\sigma(X_s^{t,x,n})|^2 d\tau\right] \\
= \; & E\left[\int_t^T |\nabla_x V^{\delta_n}(\tau,X_\tau^{t,x,\delta_n})\sigma^{\delta_n}(X_\tau^{\delta_n})-\nabla_x V^{\delta_n}(\tau,X_\tau^{t,x,\delta_n})\sigma(X_\tau^{t,x,n})|^2 d\tau\right] \\
= \; & E\left[\int_t^T |\nabla_x V^{\delta_n}(\tau,X_\tau^{t,x,\delta_n})\left(\sigma^{\delta_n}(X_\tau^{t,x,\delta_n})-\sigma(X_\tau^{t,x,n})\right)|^2 d\tau\right] \\
\leq \; & \tilde{C}E\left[\int_t^T \left\|\sigma^{\delta_n}(X_\tau^{t,x,\delta_n})-\sigma(X_\tau^{t,x,n})\right\|^2 d\tau\right] \\
\leq \; & \tilde{C}E\left[\int_t^T \left\|\sigma^{\delta_n}(X_\tau^{t,x,\delta_n})-\sigma(X_\tau^{t,x,\delta_n})\right\|^2 d\tau\right] \\
+ \; & \tilde{C}E\left[\int_t^T \left\|\sigma(X_\tau^{t,x,\delta_n})-\sigma(X_\tau^{t,x,n})\right\|^2 d\tau\right] \\
\leq \; & \tilde{C}(T-t)\delta_n^2+\tilde{C}C^2(T-t)E\left[\sup_{s\in[t,T]}|X_s^{t,x,\delta_n}-X_s^{t,x,n}|^2\right],
\end{aligned}
$$

where $\tilde{C}$ denotes the uniform bound of DV^{δ_n}. Hereafter, K will be some constant, which can be different from line to line. We obtain

$$
\begin{aligned}
& E\left[\sup_{s\in[t,T]}|Y_s^{t,x,\delta_n}-Y_s^{t,x,n}|^2\right] \\
\leq \; & E\left[|V^{\delta_n}(t,x)-V(t,x)|^2\right] \\
+ \; & (1+\gamma\epsilon)\left(\tilde{C}(T-t)\delta_n^2+\tilde{C}C^2(T-t)E\left[\sup_{s\in[t,T]}|X_s^{t,x,\delta_n}-X_s^{t,x,n}|^2\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{1+\gamma+C}{\epsilon} E\left[\sup_{s\in[t,T]} |Y_s^{t,x,\delta_n} - Y_s^{t,x,n}|^2\right] + C\epsilon E\left[\int_t^T |X_\tau^{t,x,\delta_n} - X_\tau^{t,x,n}| d\tau\right]^2 \\
&+ \quad \epsilon(T-t)\delta_n^2 \\
&\leq \quad K\delta_n^2 + KE\left[\sup_{s\in[t,T]} |X_s^{t,x,\delta_n} - X_s^{t,x,n}|^2\right] + \frac{1+\gamma+C}{\epsilon} E\left[\sup_{s\in[t,T]} |Y_s^{t,x,\delta_n} - Y_s^{t,x,n}|^2\right] \\
&\leq \quad K\delta_n^2 + \frac{1+\gamma+C}{\epsilon} E\left[\sup_{s\in[t,T]} |Y_s^{t,x,\delta_n} - Y_s^{t,x,n}|^2\right].
\end{aligned}
$$

Note that in the last line we use (5.63). Choosing ϵ such that $\frac{1+\gamma+C}{\epsilon} < 1$ and rearranging yields for any $(t,x) \in [0,T] \times \mathbb{R}^d$ and $n \in \mathbb{N}$

$$
E\left(\sup_{s\in[t,T]} |Y_s^{t,x,\delta_n} - Y_s^{t,x,n}|^2\right) \leq K\delta_n^2.
$$

Therefore (5.64) is shown.

3.) We now prove the third claim.
Similarly as in the proof of Proposition 5.2.4 we can show that for all $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\delta_n \in \mathbb{N}$ $Y_s^{t,x,\delta_n} = V^{\delta_n}(s, X_s^{t,x,\delta_n})$ for all $s \in [t,T]$. Since in Proposition 5.2.4 we also showed the convergence of V^{δ_n} to the value function V of the original control problem for $\delta_n \to 0$, we deduce from (5.63) and (5.64), that $\bar{Y}_s^{t,x} = V(s, \bar{X}_s^{t,x})$ for any $(t,x) \in [0,T] \times \mathbb{R}^d$ and $s \in [t,T]$ and in particular $\bar{Y}_T^{t,x} = k(\bar{X}_T^{t,x})$. Therefore for any $(t,x) \in [0,T] \times \mathbb{R}^d$ $(\bar{X}^{t,x}, \bar{Y}^{t,x})$ is the solution of the system

$$
\begin{cases}
d\bar{X}_s^{t,x} = b(\bar{X}_s^{t,x}, \bar{u}_s^{t,x})ds + \sigma(\bar{X}_s^{t,x})dW_s, \\
d\bar{Y}_s^{t,x} = -f(\bar{X}_s^{t,x}, \bar{u}_s^{t,x})ds + \bar{Z}_s^{t,x}dW_s, \quad s \in [t,T], \\
\bar{X}_t^{t,x} = x, \quad \bar{Y}_T^{t,x} = k(\bar{X}_T^{t,x}).
\end{cases}
\tag{5.67}
$$

On the other hand, it is well known that for the unique viscosity solution V of the Hamilton-Jacobi-Bellman equation (5.3),

$$
V(t,x) = \inf_{u\in\mathcal{U}_{\hat{\nu}}[t,T]} J(t,x,u), P-a.s., (t,x) \in [0,T] \times \mathbb{R}^d,
$$

i.e., the admissible control $\bar{u}^{t,x} \in \mathcal{U}_{\hat{\nu}}[t,T]$ is optimal for (5.67).

□

The preceding theorem shows the existence of an optimal control for the original control problem on a suitable reference stochastic system. The proof is based on an approximation of the stochastic control problem by a sequence of control problems. Whereas the original control problem has coefficients that are linear in the control variable u and therefore has a solution with bang-bang character, for the sequence of approximating control problems we use coefficients that are convex with respect to u. Therefore, we obtain besides the existence of an optimal solution a second useful result: the existence of a convergent smooth approximation of the discontinuous optimal bang-bang solution. As already mentioned, this result is very useful for the numerical solution of stochastic control problems with bang-bang controls as we will see in the following chapter.

6. Application to the Optimal Execution Problem

In this section, we will apply the results of the preceding chapter to the optimal execution problem. We begin with the formal description of the problem and will afterwards present and discuss numerical results.

6.1. Formulation of the Optimal Execution Problem with Deterministic Liquidity

We assume that there is a large investor who wants to sell X shares (measured in percentage of the average daily trading volume) of some stock during a fixed period of time $[0, T]$. We further assume that this stock is very illiquid or that the trading amount X is that big (compared to the average daily trading volume) that the trading activity will have an impact on the stock price.
Let $\nu = (\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \geq 0}, P, W)$ be a given 1-dimensional Brownian stochastic basis. We assume that without the investor's trading activity the stock price behaves like a driftless Brownian motion and that the trading impact enters the stock price in form of a drift, i.e., for $t \in [0, T]$

$$dS_t = -dD_t + \sigma dW_t, \quad S_0 = 1, \tag{6.1}$$

where D_t denotes the trading impact at time $t \in [0, T]$ and σ denotes the constant volatility of the stock.
We denote the trading speed of the investor at time $t \in [0, T]$ by u_t, such that $u_t dt$ represents for any $t \in [0, T]$ the number of traded shares during the infinitesimal time period $[t, t + dt]$. We impose the following restriction on the possible trading speed.

Assumption 6.1.1. *For every $t \in [0, T]$, u_t takes values from the set $U = [C_1, C_2]$, where $C_1, C_2 \in \mathbb{R}$ are positive constants that fulfill $C_1 \leq \frac{X}{T}$ and $C_2 \geq \frac{X}{T}$.*

Remark 6.1.2. Note that if $C_1 > \frac{X}{T}$, every possible trading strategy would induce too many traded shares. This is because if choosing at any time $t \in [0, T]$ even the lowest possible trading speed C_1, we would end up in time T with a traded volume of TC_1 shares which is more than X. Similarly, if $C_2 \geq \frac{X}{T}$, every possible trading strategy would induce too few traded shares. Obviously, if C_1 is allowed to be negative, the optimal strategy may alternate between buy and sell trades. It is clear that market impact models admitting such an alternation between selling and buying trades cannot be regarded as viable and need to be excluded. There may in fact even be legal conflicts arising from the application of such alternating strategies. We therefore assume $C_1 \geq 0$ and by doing so exclude the possibility of transaction-triggered price manipulation.

As before, we will denote the set of admissible control processes by $\mathcal{U}_\nu[0,T]$. We assume that the price impact at time $t \in [0,T]$ can be divided into a permanent impact, denoted by D_t^{perm}, and a transient impact, denoted by D_t^{trans}. The permanent trading impact is due to the informational content of the transaction, whereas the transient impact is liquidity driven and is assumed to decay exponentially over time. For any $t \in [0,T]$, we write

$$\begin{aligned} D_t &= D_t^{trans} + D_t^{perm} \\ &= \int_0^t h(u_s) e^{-\lambda(t-s)} ds + \int_0^t g(u_s) ds, \end{aligned}$$

where $g, h : U \to \mathbb{R}$ are continuously differentiable and λ represents the speed with which the temporary impact decays over time. We call this factor the *recovery rate* or alternatively the *resilience speed*. Within this work, we assume, as in Obizhaeva and Wang [2005], a constant recovery rate. Note that in recent work, there are suggestions to extend this assumption to a fixed time-dependent, deterministic recovery rate, or even a stochastic recovery rate (see Fruth [2011]).
We assume that there is no trading activity by the large investor before time $t = 0$, i.e., $u_{t+s} = 0$ for $s < -t$. Therefore, we can write the transient impact as follows.

$$D_t^{trans} = \int_{-t}^0 h(u_{t+s}) e^{\lambda s} ds = \int_{-\infty}^0 h(u_{t+s}) e^{\lambda s} ds, \quad t \in [0,T].$$

As shown in Section 4.1, the latter equation can be written equivalently as

$$dD_t^{trans} = \left(h(u_t) - \lambda D_t^{trans} \right) dt, \quad t \in [0,T]. \tag{6.2}$$

Because

$$dD_t^{perm} = g(u_t) dt, \quad t \in [0,T],$$

for any $t \in [0,T]$ the stock price process (6.1) can be written as

$$\begin{aligned} dS_t &= \left(-h(u_t) - g(u_t) + \lambda D_t^{trans} \right) dt + \sigma dW_t \\ &= \left(-h(u_t) - g(u_t) + \int_{-\infty}^0 \lambda e^{\lambda s} h(u_{t+s}) ds \right) dt + \sigma dW_t. \end{aligned} \tag{6.3}$$

As argued in the motivating Chapter 1, there are strong arguments to choose the functions g and h to be linear with respect to the trading speed. Recall that Huberman and Stanzl [2004] showed that the permanent price impact must be linear to exclude price manipulation strategies and that later Gatheral et al. [2011] proved that an exponential decay of market impact is not compatible with a nonlinear transient market impact.

Therefore, we choose in what follows

$$h(u) = k_1 u, \quad \text{and} \quad g(u) = k_2 u, \quad u \in U,$$

where k_1 and k_2 are positive constants. Consequently, for any $t \in [0,T]$ the stock price takes the form

$$\begin{aligned} dS_t &= \left(-k_1 u_t - k_2 u_t + \lambda D_t^{trans}\right) dt + \sigma dW_t \\ &= \left(-k_1 u_t - k_2 u_t + \int_{-\infty}^{0} \lambda e^{\lambda s} k_1 u_{t+s} ds\right) dt + \sigma dW_t. \end{aligned}$$

We see that the process S can be regarded as a stochastic differential equation with an exponential delay in the control variable.
Note that since it is modeled by an additive Brownian motion, the stock price may happen to become negative even without price impact. In reality, however, even very large asset positions are typically liquidated within a few days or even hours such that negative prices only occur with negligible probability. Due to the very short time horizons considered in the optimal execution problem, it is in fact quite common to choose an additive Brownian motion for S so that prices can become negative but, once again, with negligible probability. We can therefore assume that

$$E(S_T) \neq 0. \tag{6.4}$$

The trading strategy must be chosen such that all X shares are sold until time T. We denote the remaining number of shares which the investor has left to sell at time $t \in [0,T]$ by R_t and get the following expression for this process:

$$R_t = X - \int_0^t u_s ds,$$

or equivalently,

$$dR_t = -u_t dt, \quad R_0 = X.$$

Summarizing, we face an optimal control problem with the following state process.

$$\begin{cases} \begin{pmatrix} dS_t \\ dR_t \end{pmatrix} = \begin{pmatrix} -k_1 u_t - k_2 u_t + Y_t \\ -u_t \end{pmatrix} dt + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} dW_t^{(1)} \\ dW_t^{(2)} \end{pmatrix}, \quad t \in [0,T], \\ \begin{pmatrix} S_0 \\ R_0 \end{pmatrix} = \begin{pmatrix} 1 \\ X \end{pmatrix}, \end{cases} \tag{6.5}$$

where $Y_t \equiv \int_{-\infty}^{0} \lambda k_1 u_{t+s} e^{\lambda s} ds$ for any $t \in [0,T]$ and $W = (W^{(1)}, W^{(2)})$ is a 2-dimensional Brownian motion.
In Section 4.1 we showed that for an exponential delay in the control variable we are able to transform the SDDE (6.5) into a Markovian SDE with higher dimensional state

space, i.e., for any $t \in [0,T]$

$$\begin{cases} dX_t = \begin{pmatrix} dS_t \\ dR_t \\ dY_t \end{pmatrix} = \begin{pmatrix} -k_1 u_t - k_2 u_t + Y_t \\ -u_t \\ \lambda k_1 u_t - \lambda Y_t \end{pmatrix} dt + \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} dW_t^{(1)} \\ dW_t^{(2)} \\ dW_t^{(3)} \end{pmatrix}, \\ X_0 = \begin{pmatrix} S_0 \\ R_0 \\ Y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ X \\ 0 \end{pmatrix} \equiv x_0, \end{cases} \tag{6.6}$$

with the 3-dimensional standard Brownian motion $W = (W^{(1)}, W^{(2)}, W^{(3)})$.
Note that the diffusion matrix is obviously degenerate. Since the drift and the diffusion matrix fulfill the continuity and growth condition of Assumption 3.1.2, it follows that for any $(\mathcal{F}_s)_{s\geq 0}$-adapted process $u = (u_s)_{s\in[0,T]}$ there exists a unique strong solution $X^u = (S^u, R^u, Y^u)'$ to (6.6). Note that as before we will omit the superscript u, if there is no need to emphasize the solution's dependence on the specific control process.
Next, we need to formulate the cost functional. We already said that the aim is to find the optimal trading strategy such that the expected average execution price is maximized and that at time T all X shares are sold, i.e.,

$$\sup_{u\in\mathcal{U}_\nu[0,T]} E\left[\int_0^T u_t S_t dt\right] \text{ s.t. } R_T = 0.$$

Since, as in the previous chapters, we wish to formulate the objective function as a *cost* functional which needs to be minimized, we write equivalently

$$\inf_{u\in\mathcal{U}_\nu[0,T]} E\left[\int_0^T -u_t S_t dt\right] \text{ s.t. } R_T = 0. \tag{6.7}$$

The question is how to handle the constraint $R_T = 0$. One possibility could be to introduce some measurable function $k : \mathbb{R}^3 \to \mathbb{R}$ that enters the cost functional and which is chosen such that the investor has a strong incentive to sell X shares, not more and not less. With such a function, we obtain the following cost functional for our optimal execution problem.

$$J(0, x_0, u) = E\left[\int_0^T -u_t S_t dt + k(X_T)\right]. \tag{6.8}$$

The first choice of function k on which we tried the algorithm presented below was

$$k(X_T) = L R_T^2, \quad L > 0.$$

In this case, L can be interpreted as a money amount which the investor has to pay as a punishment for failing the condition $R_T = 0$.
Unfortunately, it turns out that for this proposed function the algorithm does not de-

liver usable results. But fortunately, some other information will lead to a solution: We know that the optimal control will be *static* (i.e., deterministic). Before we explain how we know this, let us explain how a static strategy differs from a *dynamic* strategy. For a static strategy, we require that the entire trade schedule must be fixed in advance before trading begins. This means that a static strategy leads to a globally optimal trading trajectory. For a dynamic strategy, we allow the trading speed to be modified in response to price motions observed during the trading period. This means that the trading strategy is allowed to get updated in "real time" using information revealed during execution. Within the optimal liquidation literature, most research was directed to finding the optimal static liquidation strategy. This is the case in for example Almgren and Chriss [1999], Almgren and Chriss [2000], or Obizhaeva and Wang [2005].
The surprising observation of Almgren and Chriss [2000] is that, under the assumption that the asset price process represents an additive random walk and that the performance functional is a linear combination of expectation and the variance of the execution cost, the statically optimal strategy is also dynamically optimal. No value is added by considering "scaling" strategies in which the execution speed changes in response to price motions. Predoiu et al. [2011] obtain similar results for a continuous setting, i.e., for the case where the stock price is formulated as an additive Brownian motion and where continuous-time trading is considered.
Let us apply the arguments of Predoiu et al. [2011] to our setting in order to show that the optimal strategy will be static. The aim of the optimal execution problem is to find among all admissible trading strategies fulfilling the constraint $R_T = 0$ the strategy u that minimizes the expected execution cost

$$E\left[\int_0^T -u_t S_t dt\right] = -E\left[\int_0^T S_t dR_t\right].$$

To compute this expectation we invoke the integration by parts formula:

$$\begin{aligned}\int_0^T S_t dR_t &= S_T R_T - S_0 R_0 - \int_0^T R_t dS_t \\ &= -S_0 X - \int_0^T R_t(-dD_t + \sigma dW_t) \\ &= -S_0 X + \int_0^T R_t dD_t - \sigma \int_0^T R_t dW_t.\end{aligned}$$

In order to compute the expression $E\left[\int_0^T R_t dW_t\right]$, we define for every $t \in [0,T]$ the process $M_t = \int_0^t R_t dW_t$. The Burkholder-Davis-Gundy inequality implies that there is some constant C such that

$$E\left[\sup_{t\in[0,T]} |M_t|\right] \leq CE\left[\int_0^T |R_s|^2 ds\right]^{\frac{1}{2}}.$$

Since the possible trading speed is bounded by the finite constants C_1 and C_2, it follows that R_t is uniformly bounded for all $t \in [0,T]$. Consequently, there exists some constant C such that

$$E\left[\sup_{t\in[0,T]} |M_t|\right] \leq C.$$

By virtue of being a local martingale, M has the property that $E[M_{\tau_n}] = 0$ for a non-decreasing sequence of stopping times $\tau_n \to T$. Finally, the dominated convergence theorem implies $E[M_T] = 0$.
It follows that the expected execution cost takes the form

$$\begin{aligned} E\left[\int_0^T -u_t S_t dt\right] &= -E\left[\int_0^T S_t dR_t\right] \\ &= S_0 X - E\left[\int_0^T R_t dD_t\right] + \sigma E\left[\int_0^T R_t dW_t\right] \\ &= S_0 X - E\left[\int_0^T R_t dD_t\right]. \end{aligned}$$

Note that in our situation the trading impact was determined by the deterministic functions h and g, that is for all $t \in [0,T]$

$$dD_t = -h(u_t) - g(u_t) + \int_{-\infty}^{0} \lambda e^{\lambda s} h(u_{t+s}) ds.$$

We see that there is no longer a source of randomness in the problem. Consequently, without loss of generality we may restrict the search for an optimal strategy to nonrandom functions of time.

Remark 6.1.3. As soon as the trading impact D is assumed to be stochastic as proposed for example in Fruth [2011], the solution will lose its property to be static. We will come back to this fact later.

We will use the knowledge that the optimal solution has to be static in order to transform the optimization problem in (6.7) into an equivalent optimization problem without constraint by introducing a Lagrange multiplier. In for example Bielecki et al. [2005] and Ji and Zhou [2006] this approach is proposed to handle a state constraint in some continuous-time portfolio-consumption problem. Before we derive the equivalent unconstrained control problem, we consider the state constraint separately.
Since the optimal control process is known to be static, we obtain that the process $R_T = X - \int_0^T u_t dt$ is deterministic. Consequently, the constraint $R_T = 0$ is equivalent to $E[R_T] = 0$. Since $E[S_T] \neq 0$ (compare (6.4)), the constraint $E[R_T] = 0$ is equivalent to

$$E[R_T S_T] = 0.$$

It follows that the problem in (6.7) can be written equivalently as

$$\inf_{u \in \mathcal{U}_\nu[0,T]} E\left[\int_0^T -u_t S_t dt\right] \text{ s.t. } E[R_T S_T] = 0. \tag{6.9}$$

The fact that u linearly enters the coefficients of the processes S and R ensures that the $\mathbb{R}$-valued functions defined by

$$\begin{aligned} u &\mapsto E\left[\int_0^T -u_s S_t^u dt\right], \\ u &\mapsto E\left[R_T^u S_T^u\right] \end{aligned}$$

are convex. Thus we can apply classical results of convex analysis (see, e.g., Luenberger [1969], Corollary, 8.3.1 and Theorem 8.4.2 for details).

Theorem 6.1.4. *Let $f : \mathbb{R}^3 \times U \to \mathbb{R}$ and $\kappa : \mathbb{R}^3 \to \mathbb{R}$ be functions such that*

$$\begin{aligned} u &\mapsto E\left[\int_0^T f(X_t^u, u_t) dt\right], \\ u &\mapsto E\left[\kappa(X_T^u)\right] \end{aligned}$$

are convex.
Let

$$\inf_{u \in \mathcal{U}_\nu[0,T]} E\left[\int_0^T f(X_t^u, u_t) dt\right] \text{ s.t. } E[\kappa(X_T^u)] = 0 \tag{6.10}$$

have a solution $\bar{u}$. Then there exists an $L \in \mathbb{R}$ such that $\bar{u}$ solves the minimization problem

$$\inf_{u \in \mathcal{U}_\nu[0,T]} E\left[\int_0^T f(X_t^u, u_t) dt + L\kappa(X_T^u)\right]. \tag{6.11}$$

Furthermore, if the minimum is attained in (6.10) by $\bar{u}$, then it is attained in (6.11) by $\bar{u}$ with $E[\kappa(X_T^{\bar{u}})] = 0$.
Conversely, suppose there exists some constant $L^0 \in \mathbb{R}$ and some admissible process u^0 such that the minimum is achieved in

$$\inf_{u \in \mathcal{U}_\nu[0,T]} E\left[\int_0^T f(X_t^u, u_t) dt + L^0\kappa(X_T^u)\right],$$

with $E\left[\kappa(X_T^{u^0})\right] = 0$, then the minimum in (6.10) is attained by u^0.

It follows that for the optimal execution problem we could consider the cost functional

$$J(0, x_0, u) = E\left[\int_0^T -u_t S_t dt + L R_T S_T\right],$$

where the constant L needs to be determined such that the resulting optimal solution $\bar{u}$

yields $R_T^{\bar{u}} = E\left[R_T^{\bar{u}} S_T^{\bar{u}}\right] = 0$. Then the preceding theorem ensures that $\bar{u}$ is the optimal solution of the original problem (6.7).

Remark 6.1.5. By writing the cost functional equivalently as

$$J(0, x_0, u) = E\left[\int_0^T -u_t S_t dt - (-LS_T) R_T\right],$$

we see that the terminal cost can be interpreted as follows. If, at terminal time T, we have shares left to sell or to buy (i.e. $R_T \neq 0$), these remaining shares can be traded at time T for the price $-LS_T$. Since we assume the stock price to be positive, this implies that L needs to be negative. Clearly, if L is too small, the trader will have an incentive to sell less than X shares during the time period $[0, T)$, because he knows that at time T he will obtain a comparatively high price. Similarly, if L is too large, the trader will have an incentive to sell more than X shares and to buy back the necessary shares cheaply at time T. Obviously, there must be one value L such that the trader will try to achieve $R_T = 0$.

It turns out that the algorithm which will be presented below delivers reasonable results for the above cost functional. Unfortunately, there is one problem with this functional: It does not fulfill Assumption 5.1.1 since the partial derivative of $k(X_T) = LR_T S_T$ with respect to R is not bounded as required. In order to apply the results from Proposition 5.2.4 and Theorem 5.3.3, we therefore need to modify the functional in a not very elegant way. The idea is to replace the function $k(X_T) = LR_T S_T$ by a smooth approximation of $LR_T(0 \vee S_T \wedge 100)$. This means that we cut off the function for values of S_T that are larger than 100 or smaller than 0. Of course the bound 100 is chosen arbitrarily. It is important that the bounds are chosen to be far away from the actual stock price such that the cutting does not really change the results with the unbounded functional. For a smooth approximation of this cut function, we use the equations

$$1_{\{x \geq x_0\}} = \lim_{n \to \infty} \left(\frac{1}{2} + \frac{1}{\pi} \arctan(n(x - x_0))\right), \quad x, x_0 \in \mathbb{R},$$

and

$$1_{\{x \leq x_0\}} = \lim_{n \to \infty} \left(\frac{1}{2} + \frac{1}{\pi} \arctan(-n(x - x_0))\right), \quad x, x_0 \in \mathbb{R}.$$

Consequently,

$$\begin{aligned}
& 0 \vee S_T \wedge 100 \\
= \; & S_T + (100 - S_T) 1_{\{S_T \geq 100\}} - S_T 1_{\{S_T \leq 0\}} \\
= \; & S_T + \lim_{n \to \infty} (100 - S_T) \left(\frac{1}{2} + \frac{1}{\pi} \arctan(n(S_T - 100))\right) \\
& \quad + \lim_{n \to \infty} \left(-S_T \left(\frac{1}{2} + \frac{1}{\pi} \arctan(-nS_T)\right)\right)
\end{aligned}$$

$$= \frac{100}{2} + \lim_{n\to\infty} \left(\left(\frac{100}{\pi} - \frac{S_T}{\pi} \right) \arctan(n(S_T - 100)) - \frac{S_T}{\pi} \arctan(-nS_T) \right).$$

Therefore, we obtain a smooth approximation of $0 \vee S_T \wedge 100$ by defining the following function for any fixed $n \in \mathbb{N}$:

$$C_{100}^n(S_T) := \frac{100}{2} + \frac{100 - S_T}{\pi} \arctan(n(S_T - 100)) - \frac{S_T}{\pi} \arctan(-nS_T). \tag{6.12}$$

Clearly, the larger the chosen value n, the better the approximation. For the numerical solution we will choose $n = 100$.
We will later need the derivative of $C_{100}^n(S_T)$ with respect to S_T which for fixed $n \in \mathbb{N}$ takes the form

$$\begin{aligned} \nabla_s C_{100}^n(S_T) = -\frac{1}{\pi} \Bigg(& \arctan(n(S_T - 100)) + \arctan(-nS_T) - \frac{n(100 - S_T)}{1 + n^2(100 - S_T)^2} \\ & - \frac{nS_T}{1 + (nS_T)^2} \Bigg). \end{aligned}$$

Using the same arguments as before, we obtain that the original constraint $R_T = 0$ is equivalent to $E[R_T C_{100}^{100}(S_T)]$. By using Theorem 6.1.4 again, we have reasons to use the following cost functional for the optimal liquidation problem:

$$J(0, x_0, u) = E\left[\int_0^T -u_t S_t dt + L R_T C_{100}^{100}(S_T) \right]. \tag{6.13}$$

In order to apply the stochastic maximum principle to this control problem, we need the terminal cost function to be convex in the state variable. Therefore, we assume $L < 0$ in what follows (which corresponds to the assumption in Remark 6.1.5). If we are able to determine a constant $L < 0$ such that the resulting trading strategy yields $R_T = E[R_T C_{100}^{100}(S_T)] = 0$, then Theorem 6.1.4 ensures that this trading strategy is optimal.
We have shown so far that the optimal execution problem can be formulated as a stochastic control problem with the state process (6.6) and the cost functional (6.13). By introducing the, unfortunately, not very intuitive smoothed cutoff function in the cost functional, we obtained a control problem whose coefficients fulfill Assumption 5.1.1. From Theorem 5.3.3 we know that an optimal control exists on a suitable reference stochastic system. It is clear that the optimal solution depends on the value of the constant L. We therefore formulate the following problem.

Problem (P_L): Minimize (6.13) with $L < 0$ being a fixed constant subject to (6.6) over $\mathcal{U}_\nu[0, T]$.

For every $L < 0$ we denote the optimal solution to Problem (P_L) by $\bar{u}^L$. We already discussed in Remark 6.1.5 that there is one value for the constant L in the cost functional such that the optimal solution yields $R_T = 0$ as desired. We denote this value with L^0

and the corresponding optimal control process by $\bar{u} := \bar{u}^{L_0}$.
Due to the linearity of the coefficients of Problem (P_L), $L < 0$, we know (see Section 4.2) that the optimal solution will be a bang-bang process and that a numerical solution is difficult or even impossible. However, using Theorem 5.3.3 again, it is possible to construct an approximating control problem which has a continuous optimal solution. This approximating control problem and the application of the stochastic maximum principle on this problem will be the topic of the next section.

6.2. The Approximating Control Problem

As in Chapter 5, we will construct the approximating control problem by substituting the degenerate diffusion matrix in (6.6) by an invertible matrix and by substituting the function f within the cost functional (6.13) by a convex function. For any $u \in \mathcal{U}_\nu[0,T]$ and any $t \in [0,T]$ we introduce the following state process for $\delta \in (0,1]$:

$$\begin{cases} dX_t^\delta = \begin{pmatrix} dS_t^\delta \\ dR_t^\delta \\ dY_t^\delta \end{pmatrix} = \begin{pmatrix} -(k_1+k_2)u_t + Y_t^\delta \\ -u_t \\ \lambda k_1 u_t - \lambda Y_t^\delta \end{pmatrix} dt + \begin{pmatrix} \sigma + \frac{\delta}{100} & 0 & 0 \\ 0 & \frac{\delta}{100} & 0 \\ 0 & 0 & \frac{\delta}{100} \end{pmatrix} \begin{pmatrix} dW_t^{(1)} \\ dW_t^{(2)} \\ dW_t^{(3)} \end{pmatrix}, \\ X_0^\delta = \begin{pmatrix} 1 \\ X \\ 0 \end{pmatrix} = x_0. \end{cases} \tag{6.14}$$

Note that in the diffusion matrix we multiplied δ with 0.01. That will stabilize the numerical results for relatively large values of δ. Since the coefficients of the SDE (6.14) fulfill the continuity and growth condition of Assumption 3.1.2, it follows that for any $(\mathcal{F}_s)_{s\geq 0}$-adapted process $u = (u_s)_{s\in[0,T]}$ there exists a unique strong solution $X^{\delta,u} = (S^{\delta,u}, R^{\delta,u}, Y^{\delta,u})'$. As before, we will omit the superscript u, if there is no need to emphasize the solution's dependence on the specific control process.
The cost functional is given for $\delta \in (0,1]$ by

$$J^\delta(0,x_0,u) = E\left[\int_0^T (-u_t S_t^\delta + \delta u_t^2)dt + L R_T^\delta C_{100}^{100}(S_T^\delta)\right], \tag{6.15}$$

where L is some positive constant and the function C_{100}^{100} is defined in (6.12).
We introduce the following control problem for any $\delta \in (0,1]$ and $L < 0$.

Problem (P_L^δ): Minimize (6.15) with $L < 0$ being a fixed constant subject to (6.14) over $\mathcal{U}_\nu[0,T]$.

For every $L < 0$ we denote the optimal solution to Problem (P_L^δ) by $\bar{u}^{\delta,L}$. In addition, we denote the value of L for which $R_T^\delta = 0$ by L^δ and the corresponding optimal control process by $\bar{u}^\delta := \bar{u}^{\delta,L^\delta}$.
Because we use a convex approximation for the linear cost functional, we do not face

the problem of a discontinuous optimal control process anymore. We will apply the stochastic maximum principle in order to derive a numerical solution of the approximating problem. For this we need the Hamiltonian function of Problem (P_L^δ) (compare equation (3.24)), which for any $\delta \in (0,1]$ and $L < 0$ takes the form

$$\begin{aligned} H^\delta(x,u,p,q) = us - \delta u^2 &- p^{(1)}(k_1 u + k_2 u - y) - p^{(2)} u + p^{(3)}(\lambda k_1 u - \lambda y) \\ &+ q^{(11)}\left(\sigma + \frac{\delta}{100}\right) + q^{(22)}\frac{\delta}{100} + q^{(33)}\frac{\delta}{100}, \end{aligned} \tag{6.16}$$

where $x = (s,r,y)^T \in \mathbb{R}^3$, $u \in U$, $p = (p^{(1)}, p^{(2)}, p^{(3)})^T \in \mathbb{R}^3$ and $q \in \mathbb{R}^{3\times 3}$ and where $q^{(ij)}$, $i,j \in \{1,2,3\}$, denotes the ij-th entry of matrix q.
For each $u \in \mathcal{U}_\nu[0,T]$, $\delta \in (0,1]$ and $L < 0$ the associated adjoint equation to Problem (P_L^δ) is the following BSDE with $t \in [0,T]$ (compare equation (3.23)):

$$\begin{cases} dp_t^{\delta,L} = \begin{pmatrix} dp_t^{\delta,L,(1)} \\ dp_t^{\delta,L,(2)} \\ dp_t^{\delta,L,(3)} \end{pmatrix} \\ \quad = \begin{pmatrix} -u_t \\ 0 \\ -p_t^{\delta,L,(1)} + \lambda p_t^{\delta,L,(3)} \end{pmatrix} dt + \begin{pmatrix} q_t^{\delta,L,(11)} & q_t^{\delta,L,(12)} & q_t^{\delta,L,(13)} \\ q_t^{\delta,L,(21)} & q_t^{\delta,L,(22)} & q_t^{\delta,L,(23)} \\ q_t^{\delta,L,(31)} & q_t^{\delta,L,(32)} & q_t^{\delta,L,(33)} \end{pmatrix} \begin{pmatrix} dW_t^{(1)} \\ dW_t^{(2)} \\ dW_t^{(3)} \end{pmatrix}, \\ p_T^{\delta,L} = \begin{pmatrix} -LR_T^\delta \cdot \nabla_s C_{100}^{100}(S_T^\delta) \\ -LC_{100}^{100}(S_T^\delta) \\ 0 \end{pmatrix}. \end{cases}$$

From Proposition 2.5.7, we know that the above BSDE has a unique solution $(p^{\delta,L}, q^{\delta,L})$. To hold the notation short, we omit a superscript that represents the dependence of the solution on u.
Consider now some fixed $\delta \in (0,1]$ and $L < 0$ and note that the coefficients of control problem (P_L^δ) fulfill Assumption 3.4.1. Recall from Theorem 3.4.3 that the maximum principle for stochastic control problems states that

- if the Hamiltonian Function H^δ is concave in u and x and the terminal cost in the cost functional is convex in the state variable, and

- if for some process $\bar{u}^{\delta,L} \in \mathcal{U}_\nu[0,T]$ with the corresponding solutions $X^{\delta,\bar{u}^{\delta,L}} = (S^{\delta,\bar{u}^{\delta,L}}, R^{\delta,\bar{u}^{\delta,L}}, Y^{\delta,\bar{u}^{\delta,L}})$, $p^{\delta,L}$ and $q^{\delta,L}$ of the above SDEs we have that $\bar{u}^{\delta,L}$ maximizes the Hamiltonian function H^δ, i.e.

  $$H^\delta(X_t^{\delta,\bar{u}^{\delta,L}}, \bar{u}_t^{\delta,L}, p_t^{\delta,L}, q_t^{\delta,L}) = \sup_{v\in\mathcal{U}} H(X_t^{\delta,\bar{u}^{\delta,L}}, v, p_t^{\delta,L}, q_t^{\delta,L})$$

  for all $t \in [0,T]$,

then $(\bar{X}^{\delta,L}, \bar{u}^{\delta,L}) := (X^{\delta,\bar{u}^{\delta,L}}, \bar{u}^{\delta,L})$ is an optimal pair (compare Definition 3.2.3) for the control problem (P_L^δ).
Therefore, in order to determine the optimal solution to the approximating control problem, we need to find the control process that takes values from the set $U = [C_1, C_2]$ and that maximizes the Hamiltonian function. Differentiating the Hamiltonian function (6.16) with respect to u and setting this expression equal to zero yields

$$s - 2\delta u - p^{(1)}(k_1 + k_2) - p^{(2)} + \lambda k_1 p^{(3)} = 0.$$

Without any restriction on the possible values which u is allowed to take, we therefore obtain the following expression that maximizes the Hamiltonian function:

$$u = \frac{1}{2\delta}\left(s - (k_1 + k_2)p^{(1)} - p^{(2)} + \lambda k_1 p^{(3)}\right).$$

With similar arguments as in Chapter 5 for the derivation of equation (5.15) we obtain that

$$u = C_1 \vee \frac{1}{2\delta}\left(s - (k_1 + k_2)p^{(1)} - p^{(2)} + \lambda k_1 p^{(3)}\right) \wedge C_2 \tag{6.17}$$

maximizes the Hamiltonian when taking into account the bounds on the control variable u.
For any $x = (s, r, y)' \in \mathbb{R}^3$ and $p = (p^{(1)}, p^{(2)}, p^{(3)}) \in \mathbb{R}^3$ we define the measurable function

$$v(x, p) = C_1 \vee \frac{1}{2\delta}\left(s - (k_1 + k_2)p^{(1)} - p^{(2)} + \lambda k_1 p^{(3)}\right) \wedge C_2.$$

Plugging this function v instead of the control process into the state variable and the adjoint equation of Problem (P_L^δ) yields for any $t \in [0, T]$

$$\begin{cases} d\bar{X}_t^{\delta,L} = \begin{pmatrix} d\bar{S}_t^{\delta,L} \\ d\bar{R}_t^{\delta,L} \\ d\bar{Y}_t^{\delta,L} \end{pmatrix} \\ \quad = \begin{pmatrix} -(k_1+k_2)v(\bar{X}_t^{\delta,L}, p_t^{\delta,L}) + \bar{Y}_t^{\delta,L} \\ -v(\bar{X}_t^{\delta,L}, p_t^{\delta,L}) \\ \lambda k_1 v(\bar{X}_t^{\delta,L}, p_t^{\delta,L}) - \lambda \bar{Y}_t^{\delta,L} \end{pmatrix} dt + \begin{pmatrix} \sigma + \frac{\delta}{100} & 0 & 0 \\ 0 & \frac{\delta}{100} & 0 \\ 0 & 0 & \frac{\delta}{100} \end{pmatrix} \begin{pmatrix} dW_t^{(1)} \\ dW_t^{(2)} \\ dW_t^{(3)} \end{pmatrix}, \\ \bar{X}_0^{\delta,L} = \begin{pmatrix} 1 \\ X \\ 0 \end{pmatrix} = x_0 \end{cases}$$

and

$$\begin{cases} dp_t^{\delta,L} = \begin{pmatrix} dp_t^{\delta,L,(1)} \\ dp_t^{\delta,L,(2)} \\ dp_t^{\delta,L,(3)} \end{pmatrix} = \begin{pmatrix} -v(\bar{X}_t^{\delta,L}, p_t^{\delta,L}) \\ 0 \\ -p_t^{\delta,L,(1)} + \lambda p_t^{\delta,L,(3)} \end{pmatrix} dt + q_t^{\delta,L} \begin{pmatrix} dW_t^{(1)} \\ dW_t^{(2)} \\ dW_t^{(3)} \end{pmatrix}, \\ p_T^{\delta,L} = \begin{pmatrix} -L\bar{R}_T^{\delta,L} \cdot \nabla_s C_{100}^{100}(\bar{S}_T^{\delta,L}) \\ -LC_{100}^{100}(\bar{S}_T^{\delta,L}) \\ 0 \end{pmatrix}. \end{cases}$$

We see that, with the help of the maximum principle, we obtain a fully coupled FBSDE. It is easy to check that the coefficients of this FBSDE fulfills the second and third condition of Assumption 2.5.15. To see that the first condition is also fulfilled, recall that the process $\bar{Y}^{\delta,L}$ represents the exponential resilience of the transient trading impact and, as shown in the previous section and in Section 4.1, has the explicit solution

$$\bar{Y}_t^{\delta,L} = \int_{-\infty}^{0} \lambda k_1 v(\bar{X}_{t+s}^{\delta,L}, p_{t+s}^{\delta,L}) e^{\lambda s} ds, \quad t \in [0,T].$$

Since the function v is bounded and since the trading activity is restricted to the bounded time interval $[0,T]$, we see that $\bar{Y}^{\delta,L}$ is bounded. Now it is easy to see that the coefficients grow at most linearly with respect to p (and also with respect to q since this process does not enter the coefficients), such that the first condition of Assumption 2.5.15 is fulfilled. Consequently, it follows from Theorem 2.5.16 that there exists a unique solution $(\bar{X}^{\delta,L}, p^{\delta,L}, q^{\delta,L})$ to the above FBSDE. This solution determines the optimal control process $\bar{u}_t^{\delta,L} := v(\bar{X}_t^{\delta,L}, p_t^{\delta,L})$, $t \in [0,T]$.

6.3. Numerical Results

For numerically solving the optimal execution problem, we fix some $L < 0$ and $\delta \in (0,1]$ and consider for any $t \in [0,T]$ the FBSDE

$$\begin{cases} d\bar{X}_t^{\delta,L} = \begin{pmatrix} d\bar{S}_t^{\delta,L} \\ d\bar{R}_t^{\delta,L} \\ d\bar{Y}_t^{\delta,L} \end{pmatrix} \\ \qquad = \begin{pmatrix} -(k_1+k_2)\bar{u}_t^{\delta,L} + \bar{Y}_t^{\delta,L} \\ -\bar{u}_t^{\delta,L} \\ \lambda k_1 \bar{u}_t^{\delta,L} - \lambda \bar{Y}_t^{\delta,L} \end{pmatrix} dt + \begin{pmatrix} \sigma + \frac{\delta}{100} & 0 & 0 \\ 0 & \frac{\delta}{100} & 0 \\ 0 & 0 & \frac{\delta}{100} \end{pmatrix} \begin{pmatrix} dW_t^{(1)} \\ dW_t^{(2)} \\ dW_t^{(3)} \end{pmatrix}, \\ \bar{X}_0^{\delta,L} = \begin{pmatrix} 1 \\ X \\ 0 \end{pmatrix} = x_0, \end{cases}$$

$$\begin{cases} dp_t^{\delta,L} = \begin{pmatrix} dp_t^{\delta,L,(1)} \\ dp_t^{\delta,L,(2)} \\ dp_t^{\delta,L,(3)} \end{pmatrix} = \begin{pmatrix} -\bar{u}_t^{\delta,L} \\ 0 \\ -p_t^{\delta,L,(1)} + \lambda p_t^{\delta,L,(3)} \end{pmatrix} dt + q_t^{\delta,L} \begin{pmatrix} dW_t^{(1)} \\ dW_t^{(2)} \\ dW_t^{(3)} \end{pmatrix}, \\ p_T^{\delta,L} = \begin{pmatrix} -L\bar{R}_T^{\delta,L} \cdot \nabla_s C_{100}^{100}(\bar{S}_T^{\delta,L}) \\ -LC_{100}^{100}(\bar{S}_T^{\delta,L}) \\ 0 \end{pmatrix}, \end{cases} \tag{6.18}$$

with

$$\bar{u}_t^{\delta,L} = C_1 \vee \frac{1}{2\delta}\left(\bar{S}_t^{\delta,L} - (k_1+k_2)p_t^{\delta,L,(1)} - p_t^{\delta,L,(2)} + \lambda k_1 p_t^{\delta,L,(3)}\right) \wedge C_2, \quad t \in [0,T]. \tag{6.19}$$

In general, a good choice to solve such an FBSDE numerically would be the algorithm presented by Bender and Zhang [2008]. This algorithm combines time discretization with an iterative scheme and in Bender and Zhang it is shown that it converges for high-dimensional coupled FBSDE under weak coupling or monotonicity conditions. However, due to the simple structure of our backward equations, it is possible to find explicit solutions to the backward part such that the numerical solution simplifies. Let us consider the three components of the backward equation separately.
By taking the conditional expectation (see also Remark 2.5.9) the first component of the above BSDE can equivalently be written as

$$p_t^{\delta,L,(1)} = E\left[-L\bar{R}_T^{\delta,L} \cdot \nabla_s C_{100}^{100}(\bar{S}_T^{\delta,L}) + \int_t^T \bar{u}_s^{\delta,L} ds | \mathcal{F}_t\right]. \tag{6.20}$$

Since for any $t \in [0,T]$ we have $\bar{R}_t^{\delta,L} = X - \int_0^t \bar{u}_s^{\delta,L} ds + \int_0^t \frac{\delta}{100} dW_s$, we get

$$\bar{R}_T^{\delta,L} - \bar{R}_t^{\delta,L} = \int_t^T \bar{u}_s^{\delta,L} ds + \int_t^T \frac{\delta}{100} dW_s, \quad t \in [0,T],$$

and equivalently,

$$\int_t^T \bar{u}_s^{\delta,L} ds = \bar{R}_T^{\delta,L} - \bar{R}_t^{\delta,L} - \int_t^T \frac{\delta}{100} dW_s, \quad t \in [0,T].$$

Consequently, (6.20) is equivalent to

$$p_t^{\delta,L,(1)} = E\left[-L\bar{R}_T^{\delta,L} \cdot \nabla_s C_{100}^{100}(\bar{S}_T^{\delta,L}) - (\bar{R}_T^{\delta,L} - \bar{R}_t^{\delta,L}) | \mathcal{F}_t\right]. \tag{6.21}$$

Since the second component of $p^{\delta,L}$ has zero drift, we obtain

$$p_t^{\delta,L,(2)} = E\left[-L C_{100}^{100}(\bar{S}_T^{\delta,L}) | \mathcal{F}_t\right]. \tag{6.22}$$

Written in integral form, the third component $p^{\delta,L}$ takes the following form.

$$p_t^{\delta,L,(3)} = \int_t^T (p_s^{\delta,L,(1)} - \lambda p_s^{\delta,L,(3)}) ds - \int_t^T q_s^{\delta,L,(3)} dW_s, \quad t \in [0,T],$$

where for every $t \in [0,T]$ $q_t^{\delta,L,(3)}$ denotes the third line of matrix $q_t^{\delta,L}$, i.e., $q_t^{\delta,L,(3)} := \left(q_t^{\delta,L,(31)}, q_t^{\delta,L,(32)}, q_t^{\delta,L,(33)}\right)$.
Applying Itô's formula to $e^{-\lambda t} p_t^{\delta,L,(3)}$ for any $t \in [0,T]$ yields

$$\begin{aligned} & e^{-\lambda t} p_t^{\delta,L,(3)} \\ = \; & -\int_t^T -e^{-\lambda s} p_s^{\delta,L,(3)} ds - \int_t^T e^{-\lambda s} dp_s^{\delta,L,(3)} \\ = \; & \int_t^T e^{-\lambda s} p_s^{\delta,L,(3)} ds - \int_t^T e^{-\lambda s}(-p_s^{\delta,L,(1)} + \lambda p_s^{\delta,L,(3)}) ds - \int_t^T e^{-\lambda s} q_s^{\delta,L,(3)} dW_s \end{aligned}$$

$$= \int_t^T e^{-\lambda s} p_s^{\delta,L,(1)} ds - \int_t^T e^{-\lambda s} q_s^{\delta,L,(3)} dW_s.$$

Multiplying both sides with $e^{\lambda t}$, $t \in [0,T]$, yields

$$p_t^{\delta,L,(3)} = \int_t^T e^{-\lambda(s-t)} p_s^{\delta,L,(1)} ds - \int_t^T e^{-\lambda(s-t)} q_s^{\delta,L,(3)} dW_s.$$

By taking the conditional expectation we obtain for any $t \in [0,T]$

$$p_t^{\delta,L,(3)} = E\left[\int_t^T e^{-\lambda(s-t)} p_s^{\delta,L,(1)} ds | \mathcal{F}_t\right]. \tag{6.23}$$

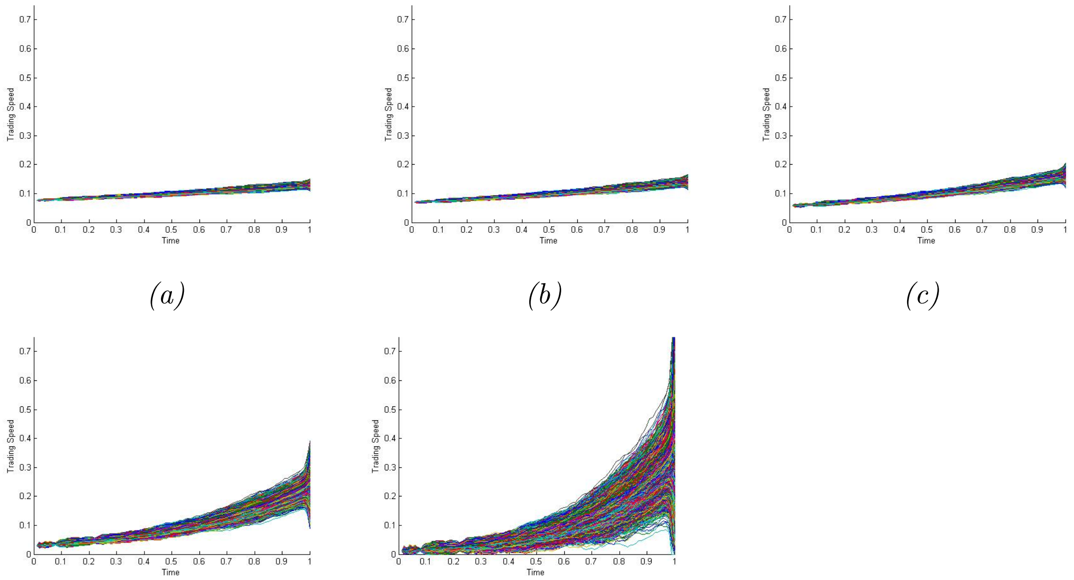
We see from equations (6.21), (6.22) and (6.23) that estimating the backward equations reduces to estimating the conditional expectations. For the estimation of the conditional expectation, we can use the approach introduced by Longstaff and Schwartz [2001]. Longstaff and Schwartz use Monte Carlo methods and show that the conditional expectation can be estimated from the cross-sectional information in the simulation by using least squares. Specifically, they regress the ex-post realized values of the backward equations from continuation on functions of the values of the state variables. The fitted value from this regression provides a direct estimate of the conditional expectation function. By estimating it for each point in time $t \in [0,T]$, we obtain a complete specification of the BSDEs solution along each path.
Our algorithm for solving Problem (P_L^δ) uses a Monte Carlo simulation and time discretization in order to simulate the stochastic differential equations. The algorithm can roughly be described in the following manner.

1. Fix some starting values for the optimal control $\bar{u}^{\delta,L}$, for example a constant trading rate of X during the whole trading period $[0,T]$.
2. Plug $\bar{u}^{\delta,L}$ into the forward equation of (6.18) and solve by discretization.
3. Use the results of the forward equations for estimating the backward equations (6.21), (6.22) and (6.23). Here we use the OLS approach by Longstaff and Schwartz [2001].
4. Use the results of the forward and backward equations for recalculating $\bar{u}^{\delta,L}$ with the help of equation (6.19).
5. Go back to step 2. Repeat several time until for each new iteration the difference between the new and the former result falls below a predefined value close to zero.

The complete algorithm in MATLAB code is documented in the Appendix.

Let us now present and discuss the simulation results. For all presented numerical results, we assume for simplicity that there is no permanent trading impact, i.e., $k_2 = 0$. The reason for this is that the permanent impact has no influence on the optimal trading

strategy since it is assumed to be proportional to the number of traded shares. For a formal proof of the fact that the permanent impact does not have an influence on the optimal trading strategy see Fruth [2011], Proposition 1.1.1. For the transient impact, we choose $k_1 = 0.5$ if another value is not explicitly specified. The aim is to sell 10% of the average daily trading volume of some share (i.e., $X = 0.1$) during one day (i.e., $T = 1$). We assume that the stock price has a daily volatility of $\sigma = 0.02$.
Let us first consider the situation where the transient impact decays very fast, i.e., $\lambda = 100$. In this case the transient impact corresponds to a temporary impact, which directly influences the current order and disappears nearly immediately after the trading activity. Figure 6.1 shows the results of the simulation for the case where there is no upper bound on the trading speed, i.e., $C_1 = 0$ and $C_2 = \infty$ and compares the results for decreasing values of δ (i.e., $\delta = 0.2, 0.15, 0.1, 0.05, 0.03$).



(a) *(b)* *(c)* *(d)* *(e)*

Figure 6.1.: *10,000 optimal trading paths for* $\lambda = 100$, $C_1 = 0$, $C_2 = \infty$ *and* $\delta = 0.2$ *(a),* $\delta = 0.15$ *(b),* $\delta = 0.1$ *(c),* $\delta = 0.05$ *(d) and* $\delta = 0.03$ *(e).*

We see, that with decreasing δ, the slope of optimal trading paths is increasing. For values of δ smaller than 0.03 the algorithm diverges. This is clear since for decreasing values of δ the optimal solution is pushed into the form of a bang-bang solution that takes, at any time $t \in [0, 1]$, either the value C_1 or C_2. Obviously, for $C_2 = \infty$, such a solution cannot exist. The simulations provide further information. In picture (a) the single paths all take a similar form. This agrees with the finding that the optimal solution is static and therefore path-independent. With decreasing values of δ, however, the variance of the single paths rises and the numerical optimal solutions depart from the path-independence. An explanation is suggested by looking at the equation that determines the optimal trading speed which takes for every $\delta \in (0,1]$, $L < 0$ and $t \in [0,T]$ the form (compare to equation (6.19))

$$\bar{u}_t^{\delta,L} = C_1 \vee \frac{1}{2\delta}\left(\bar{S}_t^{\delta,L} - (k_1+k_2)p_t^{\delta,L,(1)} - p_t^{\delta,L,(2)} + \lambda k_1 p_t^{\delta,L,(3)}\right) \wedge C_2.$$

Since the parameter δ enters by the factor $\frac{1}{\delta}$, unless the term inside the brackets equals zero (which is, at least for $t = T$, the case if the terminal values of the BSDEs $p^{\delta,L,(1)}$, $p^{\delta,L,(2)}$ and $p^{\delta,L,(3)}$ are taken into account), the optimal control takes very large values if δ approaches zero. In addition, if the bracket term takes very similar values among the single paths, these small differences will get amplified as δ decreases.
We see that it is necessary to introduce finite bounds on the control speed. Then obviously, with δ approaching zero, the solution will be pushed into a bang-bang form and it depends on the sign of the bracket term if the optimal control takes the value of the lower or the upper bound.
Figure 6.2 shows for example the numerical solution of 50.000 paths where the bounds on the trading speed are chosen to be $C_1 = 0.08$ and $C_2 = 0.12$ and $\delta = 0.001$. As expected the algorithm converges even for very small values of δ when introducing finite bounds on the control speed.

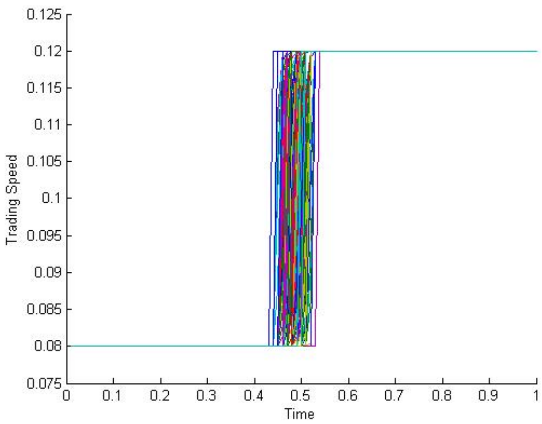


Figure 6.2.: $50,000$ *optimal trading paths for* $C_1 = 0.08$, $C_2 = 0.12$, $\delta = 0.001$ *and* $\lambda = 100$.

We see that the single paths all take the form of a bang-bang process with one change between the two extreme trading speeds around time $t = 0.5$. The time of change between the two stages occurs around the middle of the time horizon $[0,T] = [0,1]$ because C_1 and C_2 are chosen symmetrically around $X = 0.1$. Later in Figure 6.6 we will present numerical results for bounds C_1 and C_2 which are chosen asymmetrically around $X = 0.1$. We will see that the point of time where the trading speed switches between the two states changes respectively.
Obviously, we obtain the path-independence of the optimal solution which we missed for decreasing δ in the case without finite bounds on the trading speed. In order to compare

different numerical results, in the following we often depict the average of all considered paths. Figure 6.3 depicts the average of the above 50.000 paths.

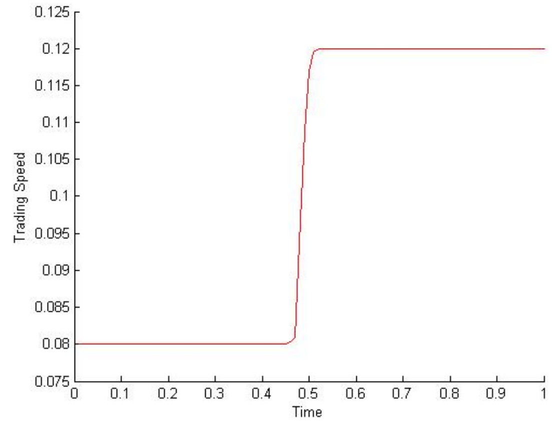


Figure 6.3.: *The average of* $50,000$ *optimal trading paths for* $C_1 = 0.08$*,* $C_2 = 0.12$*,* $\delta = 0.001$ *and* $\lambda = 100$*.*

Obviously, the faster (i.e. steeper) the change of the average trading strategy between the two extreme states C_1 and C_2, the smaller is the variance between the single paths. Since in Figure 6.3 the crossing is nearly vertical this means that despite a comparably small number of paths, all single paths look like this average trajectory. Therefore, the average is adequate to describe the behaviour of the single paths very well. In addition by the steepness of the vertical lines it gives "visual" information about the grade of path-independence of the single paths.
Figure 6.4 considers one path isolated for different values of δ and shows that this single path approaches a bang-bang solution as δ gets smaller.

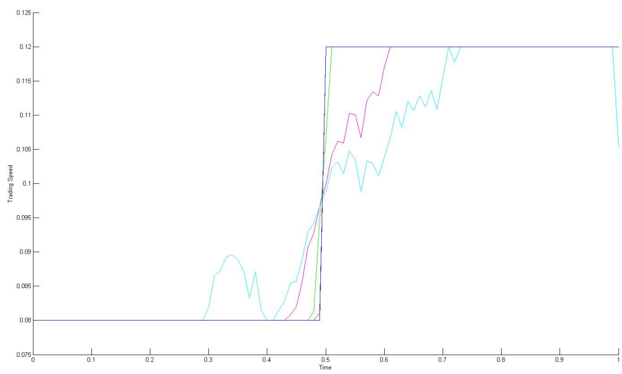


Figure 6.4.: *One single optimal trading path for* $\lambda = 100$*,* $C_1 = 0.08$*,* $C_2 = 0.12$ *and different values of* δ*. The cyan path corresponds to* $\delta = 0.1$*, the magenta path to* $\delta = 0.05$*, the green path to* $\lambda = 0.01$*, the red path - hardly to be seen below the blue path - to* $\delta = 0.001$ *and finally the blue path to* $\delta = 0.0001$*.*

The following two figures show the numerical results for different bounds C_1 and C_2 on the trading speed. In Figure 6.5 the bounds are chosen symmetrically around $X = 0.1$. We see that in each case the optimal solution is to trade approximately half of the shares with the lowest possible trading speed and the other half with the highest possible trading speed.

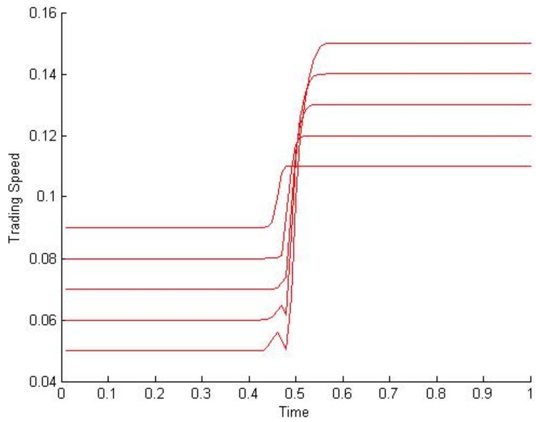


Figure 6.5.: *Average of 50,000 optimal trading paths for different values of* C_1 *and* C_2 *and for* $\lambda = 100$ *and* $\delta = 0.001$*.*

In Figure 6.6 the bounds are chosen asymmetrically around $X = 0.1$. Again we see, that the optimal strategy is split up into one part of trading with the lowest possible trading speed C_1 and the highest possible trading speed C_2. The point of switching between these two values depends on the chosen bounds.

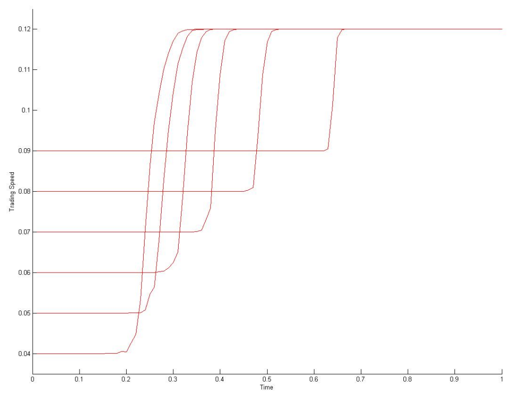
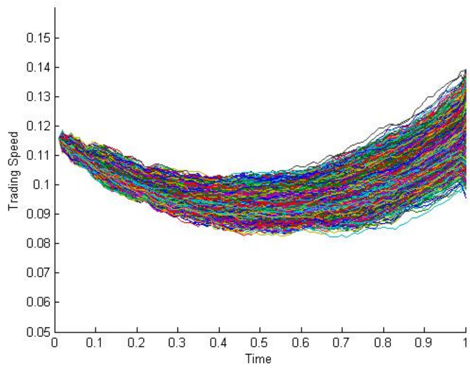
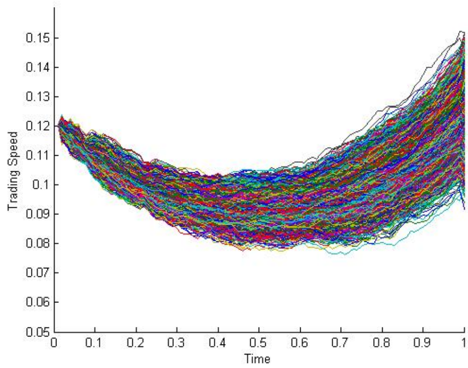


Figure 6.6.: *Average of 50,000 optimal trading paths for different values of C_1 and C_2 and for $\lambda = 100$ and $\delta = 0.001$.*

This way one could choose any other possible values for C_1 and C_2. The results agree with the existing literature since in comparable models of, for example, Almgren [2003], Almgren and Chriss [1999], Almgren and Chriss [2000] or Huberman and Stanzl [2005] it is shown that the optimal strategy is simply a constant trading speed over the whole trading period. Transferred to our case, this would imply a constant trading speed of $u_t = 0.1$ for all $t \in [0, T]$ in order to end up with a trading amount of $X = 0.1$ shares. However, this is not possible in our case, since our solution approximates a bang-bang solution that switches between two values different from 0.1. Therefore, the above numerical solution that holds the trading speed constant despite one switch between the two extreme trading levels, seems to be a reasonable "translation" of the constant trading speed into the situation of bang-bang solutions. In addition, it is shown in Gatheral et al. [2012] or Schied and Slynko [2011] that for models without transient and with only permanent impact every admissible strategy is optimal.
Let us now consider the case where the transient impact decays relatively slowly, i.e. $\lambda = 2$. This means that every trading activity has some impact on the price and this impact vanishes slowly such that also future orders are influenced. As before, we will first consider the case where there is no finite upper bound on the trading speed, i.e. $C_1 = 0$ and $C_2 = \infty$. Figure 6.7 shows the numerical result of 10.000 optimal trading paths for $\delta = 0.2$ and $\delta = 0.15$. We see that the sum of the resulting trading paths



(a)



(b)

Figure 6.7.: *10,000 optimal trading paths for $\lambda = 2$, $C_1 = 0$, $C_2 = \infty$ and $\delta = 0.2$ (a) and $\delta = 0.15$ (b).*

is U-shaped with an increasing curvature for δ getting smaller. But in both cases the variance between the single paths is too large for suggesting to call the solution static. For δ smaller than 0.15 the algorithm diverges for obviously the same reason as in the case with $\lambda = 100$.
As in the case of $\lambda = 100$, we now introduce finite bounds for C_1 and C_2. It turns out

that in this case the algorithm converges even for very small values of δ and as expected the solutions converge to a bang-bang solution. Figure 6.8 shows numerical solution of 50.000 optimal trading paths if $C_1 = 0.085$, $C_2 = 0.115$ and $\delta = 0.001$.

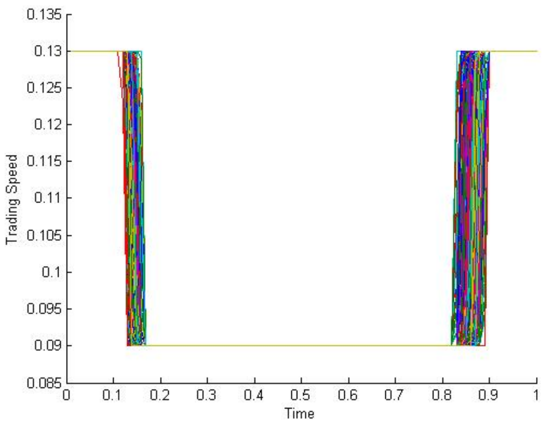


Figure 6.8.: *50.000 optimal trading paths for $C_1 = 0.085$, $C_2 = 0.115$ and $\delta = 0.001$.*

We see that now the single paths do all take a very similar U-shaped form. As before, we will in the following graphically depict the average of 50.000 solution paths. The following figures show the numerical results for different bounds on the possible trading speed. As in the case with $\lambda = 100$, we first compare in Figure 6.9 results for different choices of C_1 and C_2 symmetrically around 0.1.

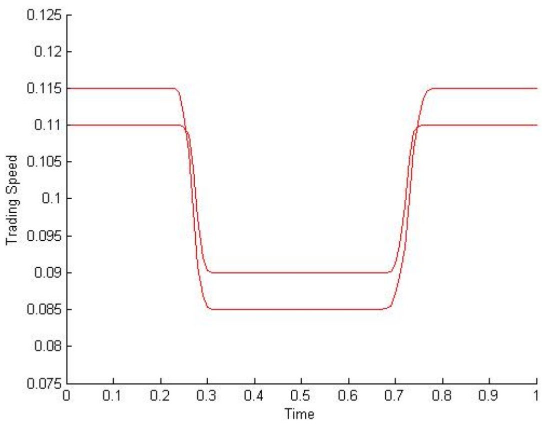


Figure 6.9.: *Average of 50,000 optimal trading paths for different values of C_1 and C_2 and for $\lambda = 2$ and $\delta = 0.001$.*

Figure 6.10 shows the result for different choices of C_1 and C_2 chosen asymmetrically around 0.1.

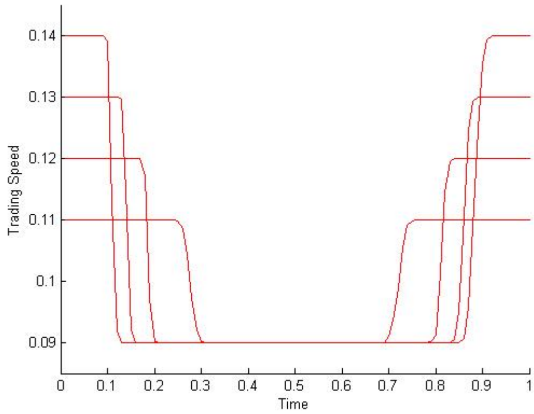


Figure 6.10.: *Average of 50,000 optimal trading paths for different values of* C_1 *and* C_2 *and for* $\lambda = 2$ *and* $\delta = 0.001$.

Unfortunately, the algorithm does not deliver convergent solutions for values of C_1 that are smaller than 0.085. We comment on this in Remark 6.3.1.
However, we see that in each of the above situations the optimal execution strategy is symmetrically divided into three parts. In the beginning and in the end, it is traded with the highest possible trading speed C_2. At intermediate times, the trading speed is the lowest possible. This result is again consistent with the results of the literature. For example, in Obizhaeva and Wang [2005], an optimal execution model is considered where continuous-time strategies as well as discrete-time impulse trades are allowed. It is shown that the optimal trading strategy consists in a large initial discrete trade followed by continuous trades with a relatively low constant speed. Finally a discrete trade of the same size as the first one occurs at the last moment T to complete the order. Also, in Gatheral et al. [2012], continuous as well as impulse trades are considered and it is shown that optimal strategies always have impulse trades at the beginning and at the end of the set T, provided that decay of the impact is convex and nonincreasing (see Gatheral et al. [2012], Theorem 2.23). Consequently, the optimal strategy is U-shaped with large trades in the beginning and end of the trading period and a smaller trading speed in between. In models that consider, as we do, only continuous-time trading, it is shown that the optimal trading strategy is U-shaped. See for example Gatheral et al. [2012] where explicit solutions for optimal execution models with linear transient impact and different decay kernels are given. For example, the optimal solution described by the remaining number of shares to be sold for a logarithmic decay is proven to be $R_t^* = \frac{2X}{\Pi} \arccos \sqrt{t}$, $t \in [0,1]$. Consequently, for any $t \in [0,1]$ the optimal trading speed is equal to $u_t^* = \frac{dR_t^*}{dt} = -\frac{X}{\Pi\sqrt{t(1-t)}}$ and graphically looks as follows.

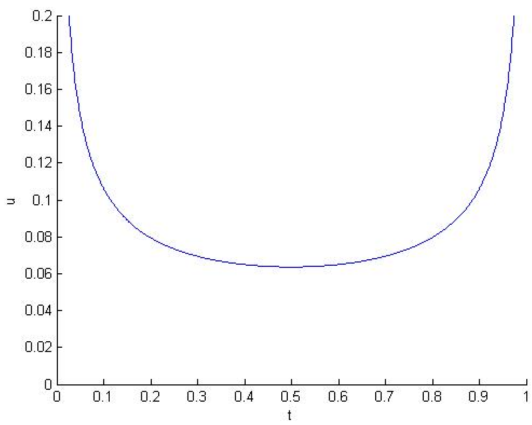


Figure 6.11.: *Optimal trading speed for a Logarithmic Decay.*

Therefore, since our solution approximates a bang-bang solution that can only take fixed given values, it seems to be reasonable that our optimal solution takes a form as in Figure 6.9 and Figure 6.10 in order to resemble a U-shaped strategy.

Remark 6.3.1. We already mentioned that in the case of a small recovery rate (i.e., $\lambda = 1$ or $\lambda = 2$) the algorithm does not deliver a good approximation of a bang-bang solution for arbitrary small values of C_1. We obtain good results for $C_1 = 0.085$, but for $C_1 = 0.08$ and even smaller values the algorithm does not converge. For $C_1 = 0.08$, for example, we can find a value L for which we get very close to the desired result $R_T = 0$, but for each iteration of the algorithm the result jumps between two values. It is still an open question why this happens.
By choosing the finite bounds C_1 and C_2, the solution is predetermined to some degree since the optimal solution converges towards a bang-bang solution that only takes one of these two values. If we are not able to choose C_1 arbitrarily below the value of 0.085, we are very restricted in the search of an optimal solution and may loose a lot of possibilities. Considering the explicit solution for the logarithmic decay in Figure 6.11 one can see that the lowest trading speed is 0.064. This means that the explicit optimal trading speed does not fall below a certain relatively large constant positive value. Therefore, the question arises, if it is for the solution approach considered here really such a big shortfall that we cannot choose values of C_1 arbitrarily small. But of course this does not allow the question of why the algorithm diverges for small values of C_1 and δ become dispensable from the technical/algorithmical point of view.

We see that the presented results agree with the results of the literature. But there are reasons why our model of optimal execution and its numerical solution is inconvenient for practical use. One reason is the search for the value L ensuring that $R_T = 0$. For every possible choice of $C_1, C_2 > 0$, this specific value L varies and needs to be determined. Another disadvantage compared to existing models is that by fixing in advance the minimal and maximal possible trading speed C_1 and C_2, our solution is predetermined to

take one of these values at any point of time $t \in [0,1]$ because we construct a bang-bang solution that only takes these two values. Depending on the choice of C_1 and C_2 the cost functional will take different values and the question that naturally arises concerns the optimal choice of parameters. One obviously faces an optimization problem within an optimization problem. This question stipulates future research.
However, other existing models do not fix the possible values for the trading speed in advance and therefore outmatch our model here. Our intention of studying the optimal execution problem, however, is not to obtain an algorithm that should be used in industrial execution algorithms. Instead, our aim is to show an example of a stochastic bang-bang problem and the possibility of its numerical solution by a smooth approximation.
The following section will extend the considered optimal execution model by introducing a time-varying liquidity.

6.4. Extensions to Optimal Execution Problems with Time-Varying Liquidity

Until now, we have assumed that the transient impact stays constant over the considered trading period of one day. Since the transient impact is liquidity driven, we have so far assumed a constant liquidity. In reality, however, one observes that liquidity exhibits strong seasonal patterns. Empirical observations by Lorenz and Osterrieder [2009], for example, show that the liquidity is U-shaped during one day. This means that on the equity market, there is less trading activity in the middle of a trading day than at the beginning. Other empirical studies such as for example Cont et al. [2010] or Malo and Pennanen [2012], find that the intraday liquidity is increasing. Recent papers such as, for example, Fruth et al. [2011] or Fruth [2011], incorporate these time-varying liquidity effects into a limit order book model. Fruth [2011] goes even further by allowing the liquidity not only to be time-varying but in addition stochastic.
In this section, we will extend our model to the situation of time-varying liquidity. This will be done by introducing into the state process a deterministic process V that describes the available trading volume for each time $t \in [0,T]$, i.e.

$$dV_t = \mathrm{VolDrift}(t) \cdot dt, \quad V_0 = v_0 \in \mathbb{R}, \quad t \in [0,T],$$

where $\mathrm{VolDrift} : [0,T] \to \mathbb{R}$ is some continuous function. For a U-shaped liquidity, we set

$$\mathrm{VolDrift}(t) = -\left(\frac{d}{(t+e)^2} - \frac{f}{(g-t)^2}\right), \quad t \in [0,T]. \tag{6.24}$$

with $d,e,f,g \in \mathbb{R}_+$ and $0 < e < 1 < g$. By setting, for example, $d = f = 0.29925$, $e = 0.3$, $g = 1.3$ and $V_0 = 1.2$, for $t \in [0,1]$ the process V takes the form

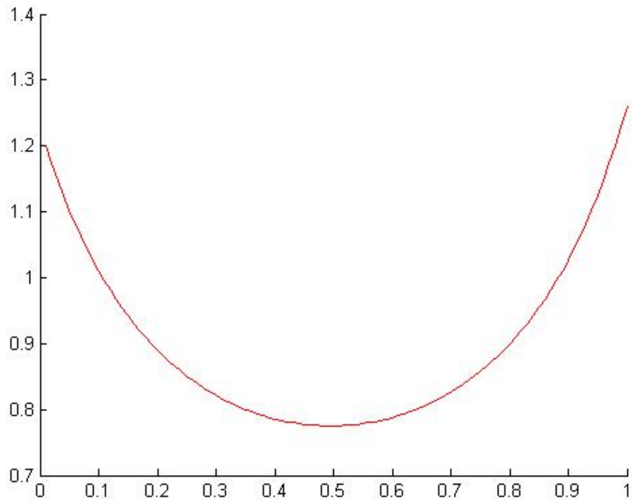


Figure 6.12.: *The process V for $d = f = 0.29925$, $e = 0.3$, $g = 1.3$ and $V_0 = 1.2$.*

For the following numerical solutions we will always use these specific values for the parameters d, f, e and g.
For an increasing liquidity, we set for all $t \in [0, T]$

$$\text{VolDrift}(t) = a, \tag{6.25}$$

where $a > 0$ is constant. If using an increasing liquidity in the following numerical solutions, we will always use $V_0 = 0.8$ and $a = 0.4$, such that we start in $t = 0$ with a liquidity of 0.8 which linearly increases until time $t = 1$ to a value of 1.2. Of course, any other non-constant positive function could be chosen for the increasing liquidity.
We will incorporate the time-dependence of the liquidity into the stock price process by dividing the constant k_1 by V_t at any time $t \in [0, T]$. This way, for $t \in [0, T]$ we obtain the state process

$$\begin{cases} dX_t = \begin{pmatrix} dS_t \\ dR_t \\ dY_t \\ dV_t \end{pmatrix} = \begin{pmatrix} -\frac{k_1}{V_t} u_t - k_2 u_t + Y_t \\ -u_t \\ \frac{\lambda k_1}{V_t} u_t - \lambda Y_t \\ \text{VolDrift}(t) \end{pmatrix} dt + \begin{pmatrix} \sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} dW_t^{(1)} \\ dW_t^{(2)} \\ dW_t^{(3)} \\ dW_t^{(4)} \end{pmatrix}, \\ \\ X_0 = \begin{pmatrix} S_0 \\ R_0 \\ Y_0 \\ V_0 \end{pmatrix} = \begin{pmatrix} 1 \\ X \\ 0 \\ v_0 \end{pmatrix}, \end{cases}$$

where $v_0 \in \mathbb{R}$ and the function VolDrift is equal to equation (6.24) if a U-shaped liquidity is considered, or alternatively, to equation (6.25) if an increasing drift is considered.

$W^{(4)}$ is a one-dimensional Brownian motion independent of $(W^{(1)}, W^{(2)}, W^{(3)})$.
The aim is, as before, to find a trading strategy that minimizes the trading costs, where the cost functional is still given by equation (6.13).
In order to solve the control problem numerically, we will construct, as in the previous section, an approximating control problem by adding to the linear function in the cost functional the term δu^2, $\delta \in (0,1]$, and by adding to the degenerate diffusion matrix the matrix δI_d, $\delta \in (0,1]$. By using the stochastic maximum principle, as in the previous section, for fixed $\delta \in (0,1]$ and $L < 0$ that needs to be solved for $t \in [0,1]$ we obtain the FBSDE

$$\begin{cases} d\bar{X}_t^{\delta,L} = \begin{pmatrix} d\bar{S}_t^{\delta,L} \\ d\bar{R}_t^{\delta,L} \\ d\bar{Y}_t^{\delta,L} \\ d\bar{V}_t^{\delta,L} \end{pmatrix} \\ \qquad = \begin{pmatrix} -\frac{k_1}{V_t}\bar{u}_t^{\delta,L} - k_2\bar{u}_t^{\delta,L} + \bar{Y}_t^{\delta,L} \\ -\bar{u}_t^{\delta,L} \\ \frac{\lambda k_1}{V_t}\bar{u}_t^{\delta,L} - \lambda\bar{Y}_t^{\delta,L} \\ \mathrm{VolDrift}(t) \end{pmatrix} dt + \begin{pmatrix} \sigma + \frac{\delta}{100} & 0 & 0 & 0 \\ 0 & \frac{\delta}{100} & 0 & 0 \\ 0 & 0 & \frac{\delta}{100} & 0 \\ 0 & 0 & 0 & \frac{\delta}{100} \end{pmatrix} \begin{pmatrix} dW_t^{(1)} \\ dW_t^{(2)} \\ dW_t^{(3)} \\ dW_t^{(4)} \end{pmatrix}, \\ \bar{X}_0^{\delta,L} = \begin{pmatrix} 1 \\ X \\ 0 \\ v_0 \end{pmatrix}, \end{cases}$$

$$\begin{cases} dp_t^{\delta,L} = \begin{pmatrix} dp_t^{\delta,L,(1)} \\ dp_t^{\delta,L,(2)} \\ dp_t^{\delta,L,(3)} \\ dp_t^{\delta,L,(4)} \end{pmatrix} = \begin{pmatrix} -\bar{u}_t^{\delta,L} \\ 0 \\ -p_t^{\delta,L,(1)} + \lambda p_t^{\delta,L,(3)} \\ -\frac{k_1}{V_t^2}\bar{u}_t^{\delta,L} p_t^{\delta,L,(1)} + \frac{\lambda k_1}{V_t^2}\bar{u}_t^{\delta,L} p_t^{\delta,L,(3)} \end{pmatrix} dt + q_t^{\delta,L} \begin{pmatrix} dW_t^{(1)} \\ dW_t^{(2)} \\ dW_t^{(3)} \\ dW_t^{(4)} \end{pmatrix}, \\ p_T^{\delta,L} = \begin{pmatrix} L\bar{R}_T^{\delta,L} \cdot \nabla_s C_{100}^{100}(\bar{S}_T^{\delta,L}) \\ LC_{100}^{100}(\bar{S}_T^{\delta,L}) \\ 0 \\ 0 \end{pmatrix}, \end{cases}$$

with

$$\bar{u}_t^{\delta,L} = C_1 \vee \frac{1}{2\delta}\left(\bar{S}_t^{\delta,L} - \left(\frac{k_1}{\bar{V}_t^{\delta,L}} + k_2\right) p_t^{\delta,L,(1)} - p_t^{\delta,L,(2)} + \frac{\lambda k_1}{\bar{V}_t^{\delta,L}} p_t^{\delta,L,(3)}\right) \wedge C_2. \qquad (6.26)$$

The last equation denotes the control process that maximizes the corresponding Hamiltonian function with respect to the restrictions on the possible trading speed and is similarly derived as equation (6.17).

Remark 6.4.1. In this work we choose FBSDEs methods to solve the stochastic control problem arising from the optimal execution problem. In the literature it is generally solved by PDE methods (recall that in Section 2.5 we pointed out the close relation between FBSDEs and quasilinear PDEs). There is one advantage of solving the optimal execution problem by FBSDE methods instead of PDE methods: We could without difficulty allow for extensions of the model. For example, one could consider a stochastic volatility of the stock price which is influenced by the liquidity process. By doing so, the dimensionality of the problem rises. For the numerical solution of FBSDEs, it is not difficult to face problems with a high dimensionality. For quasilinear PDEs, there is still the notorious "curse of dimensionality", a formidable difficulty for any numerical method. In fact, there is still no efficient numerical method for PDEs of dimension larger than three.
Another reasonable extention could be to allow for stochastic liquidity. One could, for example, consider a mean-reverting, positive Cox-Ingersoll-Ross process as done in Fruth [2011]. As explained we would have no technical problems to handle this higher dimensionality when solving the corresponding FBSDE numerically. We would rather obtain some other problem when introducing stochastic liquidity: In Remark 6.1.3 we argued that the optimal solution is no longer static.
But the cost functional we chose so far requires the optimal solution to be static. Recall that in the beginning of this chapter we prove that we can incorporate, by means of Theorem 6.1.4, a constraint of the form $E(R_T) = 0$ into the cost functional by modifying it with some terminal condition that contains an undetermined constant L. Theorem 6.1.4 then implies that there is some value L such that the optimal solution of the modified cost functional ensures $E(R_T) = 0$. If the optimal solution is static, we obtain as desired $R_T = 0$.
Consequently, if we introduce stochastic liquidity, we face problems with the choice of the cost functional. Finding an alternative cost functional is left to future research.

We now apply an algorithm similar to that of the previous section in order to solve this FBSDE. The corresponding MATLAB code is to be found in the Appendix. Note that since the new fourth component of the adjoint equation $p^{\delta,L,(4)}$ does neither enter the coefficients of the above FBSDE nor the expression (6.26) of the optimal control process, there is no need to simulate this process.
Intuitively, it is clear that in the case of the U-shaped liquidity the optimal strategy should trade more shares in the beginning and in the end of the trading period than in the case with a constant liquidity. It turns out that the numerical results confirm this intuition. In the following figure we consider the average of 50,000 optimal trading paths. Figure 6.13 compares, for example, the average optimal solution of the model with U-shaped liquidity with the average optimal solution of the constant liquidity model for the case $\lambda = 100$, $C_1 = 0.07$, $C_2 = 0.13$ and $k_1 = 0.7$.

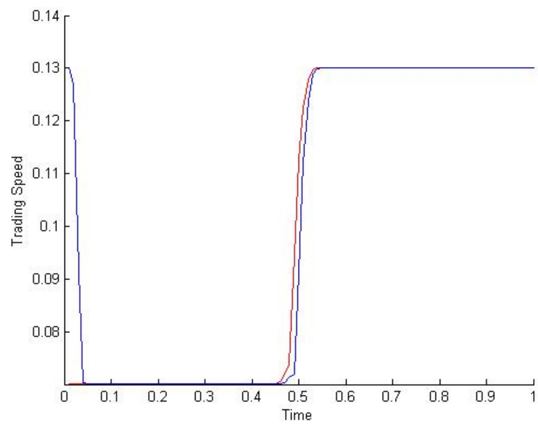


Figure 6.13.: *Average of 50,000 optimal trading paths for* $\lambda = 100$*,* $\delta = 0.001$*,* $C_1 = 0.07$*,* $C_2 = 0.13$ *and* $k_1 = 0.7$*. The red curve corresponds to the optimal solution with a constant liquidity, the blue line to the optimal solution with a U-shaped liquidity.*

We see that in the case with the U-shaped liquidity the optimal solution also takes a bang-bang form. But in contrast to the solution with constant liquidity, where the trading strategy is divided equally into one period with a trading speed C_1 and into one trading period with trading speed C_2, it is now optimal to trade in the beginning as well as in the end of the trading period with the maximal possible trading speed C_2 in order to profit from the high available liquidity.
Figure 6.14 compares the results of the U-shaped liquidity model with the constant liquidity model for the case $\lambda = 2$, $C_1 = 0.09$, $C_2 = 0.13$ and $k_1 = 0.7$. We see that the solutions are nearly identical. An explanation is, of course, that the optimal solution for a constant liquidity is already U-shaped and therefore profits from the high liquidity occur in the beginning and at the end of the trading period.

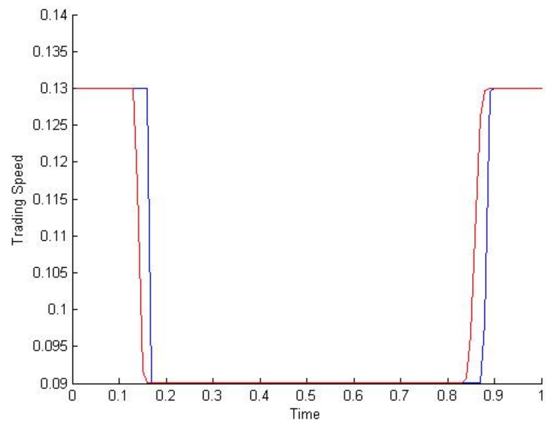


Figure 6.14.: *Average of 50,000 optimal trading paths for* $\lambda = 2$*,* $\delta = 0.001$*,* $C_1 = 0.09$*,* $C_2 = 0.13$ *and* $k_1 = 0.7$*. The red curve corresponds to the optimal solution with a constant liquidity, the blue one to the optimal solution with a U-shaped liquidity.*

Let us now compare selected results of the increasing liquidity model with the results of the constant liquidity model. As before, we consider the case where $\lambda = 2$, $C_1 = 0.09$, $C_2 = 0.13$ and $k_1 = 0.7$. Figure 6.15 shows the numerical simulations of the optimal solutions.

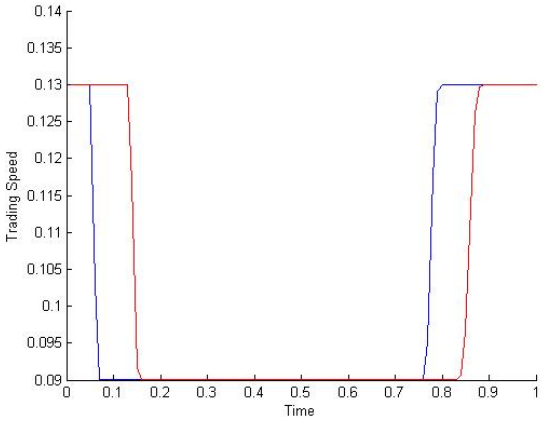


Figure 6.15.: *Average of 50,000 optimal trading paths for $\lambda = 2$, $\delta = 0.001$, $C_1 = 0.09$, $C_2 = 0.13$ and $k_1 = 0.7$. The red curve corresponds to the optimal solution with constant liquidity, the blue one to the optimal solution with a U-shaped liquidity.*

In both cases the optimal solutions are U-shaped. In the case of constant liquidity the proportion of shares that is sold with the highest possible trading speed C_2 is equally divided between trading intervals at the beginning and at the end of the trading period. This is not the case if liquidity is assumed to be increasing over time. Then, more shares are traded at the end of the trading period in order to benefit from the high liquidity.
In Figure 6.16 the optimal solutions for the model with increasing liquidity and with constant liquidity are compared in the situation in which the trading impact has a large recovery rate, i.e. $\lambda = 100$. The other parameters are chosen again to be $C_1 = 0.07$, $C_2 = 0.13$ and $k_1 = 0.7$.

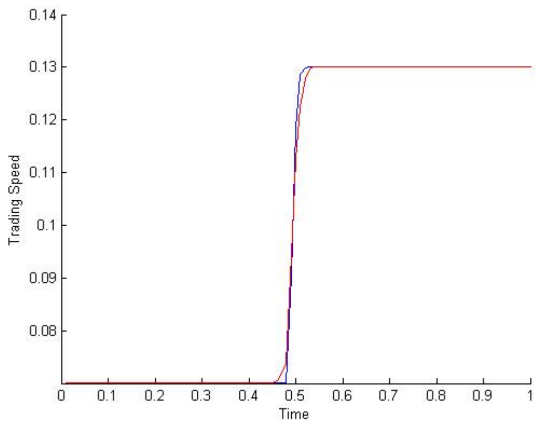


Figure 6.16.: *Average of 50,000 optimal trading paths for* $\lambda = 100$*,* $\delta = 0.001$*,* $C_1 = 0.07$*,* $C_2 = 0.13$ *and* $k_1 = 0.7$*. The red curve corresponds to the optimal solution with constant liquidity, the blue one to the optimal solution with U-shaped liquidity.*

Obviously, the results of the two different models look very similar. An explanation for this is that the optimal solution for the constant liquidity already favors sales in the second half of the trading period with highest possible trading speed C_2. Since this is also the time interval with maximal liquidity, it seems to be reasonable that this solution resembles the optimal solution for the model with increasing liquidity.

6.5. Concluding Remarks

In the preceding sections, we present a numerical method for the smooth approximation of a stochastic bang-bang problem applied to the optimal execution problem. Although the algorithm applied delivers a number of useful and good results, there are still tasks for future research.

One major task is to look for an alternative formulation of the cost functional for which the algorithm produces usable results. Such an alternative cost functional should imply that exactly the desired amount of shares is traded without requiring to search for the unknown parameter L that ensures that $R_T = 0$. In addition it should not demand the optimal solution to be static. If such an alternative functional is determined, and the algorithm converges, the dimension of the model could be extended to a more realistic and complex scenario. To this end, one could introduce for instance stochastic liquidity. The passage to higher dimensions would be supported by the fact that we work with BSDEs. Another topic for further research is the question, how to choose the finite bounds C_1 and C_2 on the trading speed optimally.

As already pointed out, do all the here named open questions give reason why existing alternative models for the solution of the optimal execution problem outmatch the solution presented here. But we successfully presented an example of the numerical solution of a stochastic-bang problem by smoothing methods which was the intention of this Chapter.

A. Appendix

A.1. Matlab Code for Section 6.3

```
clear
hold on
format long


%Choose minimal and maximal bound for the trading speed
c1=0.08; c2=0.12;

%Choose number of iterations and a value for delta out of the set (0,1]
NumberOfIterations=50;
delta=0.001;

%Choose a value for the Lagrangian L
Lagrange=-0.986375;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%The following lines show for different szenarios the values of the
variable "Lagrangian" which yield R_T=0.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%lambda=2

%no bound for c2 (resp. a very large value), i.e. c1=0; c2=10
%a=0; n=100; m=10000; k1=0.5;
%delta=0.2; Lagrange=-0.95185
%a=0; n=100; m=10000; k1=0.5;
%delta=0.15; Lagrange=-0.96231

%-----------------------------

%c1=0.09; c2=0.11;
%a=0; n=100; m=50000; k1=0.5;
%delta=0.001; Lagrange=-0.99168;

%-----------------------------
```

```
%c1=0.09; c2=0.12;
%a=0; n=100; m=50000; k1=0.5;
%delta=0.001; Lagrange=-0.9936;

%----------------------------

%c1=0.09; c2=0.13;
%a=0; n=100; m=50000; k1=0.5;
%delta=0.001; Lagrange=-0.9948;

%a=0; n=100; m=50000; k1=0.7;
%delta=0.001; Lagrange=-0.99275;

%----------------------------

%c1=0.09; c2=0.14;
%a=0; n=100; m=50000; k1=0.5;
%delta=0.001; Lagrange=-0.995636;

%----------------------------

%c1=0.085;c2=0.115;
%a=0; n=100; m=50000; k1=0.5;
%delta=0.001; Lagrange=-0.99208;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%lambda=100

%no bound for c2 (resp. a very large value), i.e. c1=0; c2=10
%a=0; n=100; m=10000; k1=0.5;
%delta=0.2; Lagrange=-0.9479
%a=0; n=100; m=10000; k1=0.5;
%delta=0.15; Lagrange=-0.9577
%a=0; n=100; m=10000; k1=0.5;
%delta=0.1; Lagrange=-0.9672
%a=0; n=100; m=10000; k1=0.5;
%delta=0.05; Lagrange=-0.97576
%a=0; n=100; m=10000; k1=0.5;
%delta=0.03; Lagrange=-0.978183

%----------------------------

%c1=0.09;c2=0.11;
%a=0; n=100; m=50000; k1=0.5;
```

```
%delta=0.001; Lagrange=-0.9869;

%----------------------------

%c1=0.08;c2=0.12;
%a=0; n=100; m=50000; k1=0.5;
%delta=0.001; Lagrange=-0.986375;

%----------------------------

%c1=0.09;c2=0.12;
%a=0; n=100; m=50000; k1=0.5;
%delta=0.001; Lagrange=-0.99026;

%----------------------------

%c1=0.07;c2=0.12;
%a=0; n=100; m=50000; k1=0.5;
%delta=0.001; Lagrange=-0.98298;

%----------------------------

%c1=0.06;c2=0.12;
%a=0; n=100; m=50000; k1=0.5;
%delta=0.001; Lagrange=-0.98253;

%----------------------------

%c1=0.05;c2=0.12;
%a=0; n=100; m=50000; k1=0.5;
%delta=0.001; Lagrange=-0.98147;

%----------------------------

%c1=0.04;c2=0.12;
%a=0; n=100; m=50000; k1=0.5;
%delta=0.001; Lagrange=-0.98066;

%----------------------------

%c1=0.07;c2=0.13;
%a=0; n=100; m=50000; k1=0.5;
%delta=0.001; Lagrange=-0.98551;
```

```
%a=0; n=100; m=50000; k1=0.7;
%delta=0.001; Lagrange=-0.9798;

%----------------------------

%c1=0.06;c2=0.14;
%a=0; n=100; m=50000; k1=0.5;
%delta=0.001; Lagrange=-0.98459;

%----------------------------

%c1=0.05;c2=0.15;
%a=0; n=100; m=50000; k1=0.5;
%delta=0.001; Lagrange=-0.98368;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

tic

%Number of shares to be bought as a fraction of average daily volume
X=0.1;

%Parameters of Forward Equations
S0=1; sigma1=0.02;
Y0=0;
R0=X;

%Number of Simulations and Step Width
m=50000; n=100; T=1; dt=T/n; t=dt:dt:T;

%Parameters for Transient Impact
k1=0.5; lambda=100;

%Parameter for Permanent Impact
k2=0;


%Euler Scheme for Forward Equations
randn('state',1)
dW1=sqrt(dt)*randn(m,n);
W1=cumsum(dW1,2);
randn('state',2)
dW2=sqrt(dt)*randn(m,n);
W2=cumsum(dW2,2);
```

```
randn('state',3)
dW3=sqrt(dt)*randn(m,n);
W3=cumsum(dW3,2);
S=S0*ones(m,n);
Y=Y0*ones(m,n);
R=R0*ones(m,n);
Rest=R0*ones(m,n);

%Starting value for optimal trading strategy
u=zeros(m,n);
for w=1:n
    u(:,w)=n/(n-1)*X;
end

%Starting Value for the Backward Equations
p1=0*ones(m,n);p2=0*ones(m,n);p3=0*ones(m,n);
%Introduce other helping and necessary variables;
HelpingVariable=zeros(m,1);HelpingVariable2=zeros(m,1);HelpingVariable3
=zeros(m,1);
TradingCosts=zeros(NumberOfIterations,1);
TotalRest=zeros(NumberOfIterations,1);
CostFunctional=zeros(NumberOfIterations,1);
Penalty=zeros(NumberOfIterations,1);

%First Iteration
k=1;
    Drift=zeros(m,n);
    for i=2:n
        Y(:,i)=Y(:,i-1)+(lambda*k1*u(:,i)-lambda*Y(:,i-1))*dt
               +0.01*delta*dW3(:,i-1);
        Drift(:,i)=(-k2*u(:,i)-k1*u(:,i)+Y(:,i));
        S(:,i)=S(:,i-1)+Drift(:,i)*dt+(sigma1+0.01*delta)*dW1(:,i-1);
        R(:,i)=R(:,i-1)-u(:,i)*dt+0.01*delta*dW2(:,i-1);
        Rest(:,i)=Rest(:,i-1)-u(:,i)*dt;

        HelpingVariable=cumsum(-S(:,i).*u(:,i)*dt,1);
        HelpingVariable2=cumsum((-S(:,i).*u(:,i)+
                          delta*u(:,i).^2)*dt,1);
        TradingCosts(k)=TradingCosts(k)+1/m*HelpingVariable(m);
    end

    sum2=cumsum(Rest(:,n));
    TotalRest(k)=1/m*sum2(m);

    HelpingVariable3=cumsum(-Lagrange*R(:,n).*(50+(1/pi*(100-
```

```
                S(:,n)).*atan(100*(S(:,n)-100))-1/pi*S(:,n).*
                atan(-100*S(:,n)))),1);
    Penalty(k)=1/m*HelpingVariable3(m);

    CostFunctional(k)=TradingCosts(k)+Penalty(k);

%Further Iterations

for k=2:NumberOfIterations

    sum=zeros(m,n);

    %Adjoint Equations
    Koeff1=zeros(2,n);
    Koeff2=zeros(2,n);
    Koeff3=zeros(2,n);
    x=zeros(m,2);

    %Terminal Values of the Adjoint Equations
    p1(:,n)=-Lagrange*(-1/pi*(atan(100*(S(:,n)-100))+atan(-100*S(:,n))-
            (100*(100-S(:,n)))./(1+100.^2*(100-S(:,n)).^2)-
            100*S(:,n)./(1+100^2*S(:,n).^2))).*R(:,n);

    p2(:,n)=-Lagrange*(50+(1/pi*(100-S(:,n)).*atan(100*(S(:,n)-100))-
            1/pi*S(:,n).*atan(-100*S(:,n))));


    for i=1:n-1
        j=n-i;

        %Basis Function
        x=[ones(m,1)  S(:,j)];

        %Estimates of the Adjoint Equations
        y1=p1(:,n)-(R(:,n)-R(:,j));
        Koeff1(:,j)=pinv(x,0.0001)*y1;
        p1(:,j)=1/1*x*Koeff1(:,j);

        y2=p2(:,n);
        Koeff2(:,j)=pinv(x,0.0001)*y2;
        p2(:,j)=1/1*x*Koeff2(:,j);

        sum=zeros(m,1);
        for q=1:i
            sum=sum+exp(-lambda*q*dt)*p1(:,j+q)*dt;
```

```
        end
        y3=1*sum;
        Koeff3(:,j)=pinv(x,0.00001)*y3;
        p3(:,j)=x*Koeff3(:,j);
    end

    for i=1:n
        u(:,i)=max(min(1/(2*delta)*(S(:,i)-(k1+k2)*p1(:,i)-
              p2(:,i)+lambda*k1*p3(:,i)),c2),c1);
    end

    %Forward Equations
    Drift=zeros(m,n);
    for i=2:n
        Y(:,i)=Y(:,i-1)+(lambda*k1*u(:,i)-lambda*Y(:,i-1))*dt
               +0.01*delta*dW3(:,i-1);
        Drift(:,i)=(-k2*u(:,i)-k1*u(:,i)+Y(:,i));
        S(:,i)=S(:,i-1)+Drift(:,i)*dt+(sigma1+0.01*delta)*dW1(:,i-1);
        R(:,i)=R(:,i-1)-u(:,i)*dt+0.01*delta*dW2(:,i-1);
        Rest(:,i)=Rest(:,i-1)-u(:,i)*dt;

        HelpingVariable=cumsum(-S(:,i).*u(:,i)*dt,1);
        HelpingVariable2=cumsum((-S(:,i).*u(:,i)+
                         delta*u(:,i).^2)*dt,1);
        TradingCosts(k)=TradingCosts(k)+1/m*HelpingVariable(m);
    end

    sum2=cumsum(Rest(:,n));
    TotalRest(k)=1/m*sum2(m);

    HelpingVariable3=cumsum(-Lagrange*R(:,n).*(50+(1/pi*(100-
                     S(:,n)).*atan(100*(S(:,n)-100))-
                     1/pi*S(:,n).*atan(-100*S(:,n)))),1);
    Penalty(k)=1/m*HelpingVariable3(m);

    CostFunctional(k)=TradingCosts(k)+Penalty(k);
End

sum=cumsum(u,1);

%Plot Average Trading Speed
plot(t,1/m*sum(m,:),'red');
   xlabel('Time')
   ylabel('Trading Speed')
```

```
%Plot Single Paths of Optimal Trading Speed
%plot(t,u')
%   xlabel('Time')
%   ylabel('Trading Speed')

TradingCosts
CostFunctional
TotalRest
V=Var(R(:,n))
toc
```

A.2. Matlab Code for Section 6.4

```
clear
hold on
format long


%Choose minimal and maximal bound for the trading speed
c1=0.07; c2=0.13;

%Choose number of iterations and a value for delta out of the set
(0,1]
NumberOfIterations=50;
delta=0.001;

%Choose a value for the Lagrangian L
Lagrange=-0.98298;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%The following lines show for different szenarios the values of the
variable "Lagrangian" which yield R_T=0.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%U-SHAPED LIQUIDITY
%V0=1.2; d=0.2925; f=0.2925; e=0.3; g=1.3;

%lambda=2, m=10000; n=100

%c1=0.09;c2=0.13;
%delta=0.001; k1=0.7; m=50000; Lagrange=-0.9931

%-------------------------------------------------

%lambda=100, m=10000; n=100

%c1=0.07;c2=0.13;
%delta=0.001; k1=0.7; m=50000; Lagrange=-0.97881

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%INCREASING LIQUIDITY
%V0=0.8; a=0.4;

%lambda=2, n=100

%c1=0.09;c2=0.13;
%delta=0.001; k1=0.7; m=50000; Lagrange=-0.9912
```

```
%--------------------------------------------------

%lambda=100, m=10000; n=100

%c1=0.07;c2=0.13;
%delta=0.001; k1=0.7; m=50000; Lagrange=-0.98298

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

tic

%Number of shares to be bought as a fraction of average daily volume
X=0.1;

%Parameters of Forward Equations
S0=1.2; sigma1=0.02;
%Parameters for U-shaped Liquidity
%V0=1.2; d=0.2925; f=0.2925; e=0.3; g=1.3;
%Parameters for increasing Liquidity
V0=0.8; a=0.4;
Y0=0;
R0=X;

%Number of Simulations and Step Width
m=50000; n=100; T=1; dt=T/n; t=dt:dt:T;

%Parameters for Transient Impact
k1=0.5; lambda=100;

%Parameter for Permanent Impact
k2=0;

%Euler Scheme for Forward Equations
randn('state',1)
dW1=sqrt(dt)*randn(m,n);
W1=cumsum(dW1,2);
randn('state',2)
dW2=sqrt(dt)*randn(m,n);
W2=cumsum(dW2,2);
randn('state',3)
dW3=sqrt(dt)*randn(m,n);
W3=cumsum(dW3,2);
dW4=sqrt(dt)*randn(m,n);
W4=cumsum(dW3,2);
S=S0*ones(m,n);
V=V0*ones(m,n);
Y=Y0*ones(m,n);
R=R0*ones(m,n);
Rest=R0*ones(m,n);
```

```
%Starting value for optimal trading strategy
u=zeros(m,n);
for w=1:n
    u(:,w)=n/(n-1)*X;
end

%Starting Value for the Backward Equations
p1=0*ones(m,n);p2=0*ones(m,n);p3=0*ones(m,n);

%Introduce other helping and necessary variables;
HelpingVariable=zeros(m,1);HelpingVariable2=zeros(m,1);HelpingVariab
le3=zeros(m,1);
TradingCosts=zeros(NumberOfIterations,1);
TotalRest=zeros(NumberOfIterations,1);
CostFunctional=zeros(NumberOfIterations,1);
Penalty=zeros(NumberOfIterations,1);

%First Iteration
k=1;
    Drift=zeros(m,n);
    for i=2:n
        %U-shaped liquidity$
        %V(:,i)=V(:,i-1)-(d/(i*dt+e)^2-f/(g-i*dt)^2)*dt+
                0.01*delta*dW4(:,i-1);
        %Increasing liquidity$
        V(:,i)=V(:,i-1)+a*dt+0.01*delta*dW4(:,i-1);
        Y(:,i)=Y(:,i-1)+(lambda*k1*u(:,i)./V(:,i)-
               lambda*Y(:,i-1))*dt+0.01*delta*dW3(:,i-1);
        Drift(:,i)=(-k2*u(:,i)-k1*u(:,i)./V(:,i)+Y(:,i));
        S(:,i)=S(:,i-1)+Drift(:,i)*dt+(sigma1+0.01*delta)*dW1(:,i-
1);
        R(:,i)=R(:,i-1)-u(:,i)*dt+0.01*delta*dW2(:,i-1);
        Rest(:,i)=Rest(:,i-1)-u(:,i)*dt;

        HelpingVariable=cumsum(-S(:,i).*u(:,i)*dt,1);
        HelpingVariable2=cumsum((-S(:,i).*u(:,i)+
                         delta*u(:,i).^2)*dt,1);
        TradingCosts(k)=TradingCosts(k)+1/m*HelpingVariable(m);
    end

    sum2=cumsum(Rest(:,n));
    TotalRest(k)=1/m*sum2(m);

    HelpingVariable3=cumsum(-Lagrange*R(:,n).*(50+(1/pi*(100-
                  S(:,n)).*atan(100*(S(:,n)-100))-1/pi*S(:,n).*
                  atan(-100*S(:,n)))),1);
    Penalty(k)=1/m*HelpingVariable3(m);
```

```
    CostFunctional(k)=TradingCosts(k)+Penalty(k);

%Further Iterationens

for k=2:NumberOfIterations

    sum=zeros(m,n);

    %Adjoint Equations
    Koeff1=zeros(2,n);
    Koeff2=zeros(2,n);
    Koeff3=zeros(2,n);
    x=zeros(m,2);

    %Terminal Values of the Adjoint Equations
    p1(:,n)=-Lagrange*(-1/pi*(atan(100*(S(:,n)-100))+
             atan(-100*S(:,n))-(100*(100-S(:,n)))./
             (1+100.^2*(100-S(:,n)).^2)-100*S(:,n)./
             (1+100^2*S(:,n).^2))).*R(:,n);

    p2(:,n)=-Lagrange*(50+(1/pi*(100-S(:,n)).*
            atan(100*(S(:,n)-100))-1/pi*S(:,n).*atan(-100*S(:,n))));

    for i=1:n-1
        j=n-i;

        %Basis Function
        x=[ones(m,1)  S(:,j)];

        %Estimates of the Adjoint Equations
        y1=p1(:,n)-(R(:,n)-R(:,j));
        Koeff1(:,j)=pinv(x,0.0001)*y1;
        p1(:,j)=1/1*x*Koeff1(:,j);

        y2=p2(:,n);
        Koeff2(:,j)=pinv(x,0.0001)*y2;
        p2(:,j)=1/1*x*Koeff2(:,j);


        sum=zeros(m,1);
        for q=1:i
            sum=sum+exp(-lambda*q*dt)*p1(:,j+q)*dt;
        end
        y3=1*sum;
        Koeff3(:,j)=pinv(x,0.00001)*y3;
        p3(:,j)=x*Koeff3(:,j);

    end
```

```
    for i=1:n
        u(:,i)=max(min(1/(2*delta)*(S(:,i)-k1*p1(:,i)./V(:,i)-
               k2*p1(:,i)-p2(:,i)+lambda*k1*p3(:,i)./V(:,i)),
               c2),c1);
    end

    %Forward Equations
    Drift=zeros(m,n);
    for i=2:n
        %U-shaped liquidity
        %V(:,i)=V(:,i-1)-(d/(i*dt+e)^2-f/(g-i*dt)^2)*dt
               +0.01*delta*dW4(:,i-1);
        %Increasing liquidity
        V(:,i)=V(:,i-1)+a*dt+0.01*delta*dW4(:,i-1);
        Y(:,i)=Y(:,i-1)+(lambda*k1*u(:,i)./V(:,i)-
               lambda*Y(:,i-1))*dt+0.01*delta*dW3(:,i-1);
        Drift(:,i)=(-k2*u(:,i)-k1*u(:,i)./V(:,i)+Y(:,i));
        S(:,i)=S(:,i-1)+Drift(:,i)*dt+(sigma1+0.01*delta)*
               dW1(:,i-1);
        R(:,i)=R(:,i-1)-u(:,i)*dt+0.01*delta*dW2(:,i-1);
        Rest(:,i)=Rest(:,i-1)-u(:,i)*dt;

        HelpingVariable=cumsum(-S(:,i).*u(:,i)*dt,1);
        HelpingVariable2=cumsum((-S(:,i).*u(:,i)+
                         delta*u(:,i).^2)*dt,1);
        TradingCosts(k)=TradingCosts(k)+1/m*HelpingVariable(m);
    end

    sum2=cumsum(Rest(:,n));
    TotalRest(k)=1/m*sum2(m);

    HelpingVariable3=cumsum(-Lagrange*R(:,n).*(50+(1/pi*
                     (100-S(:,n)).*atan(100*(S(:,n)-100))-
                     1/pi*S(:,n).*atan(-100*S(:,n)))),1);
    Penalty(k)=1/m*HelpingVariable3(m);

    CostFunctional(k)=TradingCosts(k)+Penalty(k);
end

sum=cumsum(u,1);

%Plot Average Trading Speed
plot(t,1/m*sum(m,:),'b');
   xlabel('Time')
   ylabel('Trading Speed')

%Plot Single Paths of Optimal Trading Speed
%plot(t,u')
%   xlabel('Time')
```

```
%   ylabel('Trading Speed')

TradingCosts
CostFunctional
TotalRest
V=Var(R(:,n))
toc
```

Symbols and Notation

The following notation is frequently used.

- $\mathbb{Q}$ - the set of all rational numbers
- $\mathbb{N}$ - the set of all natural numbers
- $\mathbb{R}^n$ - n-dimensional real Euclidean space
- $\mathbb{R}^{n\times m}$ - the space of all $n \times m$ real matrices
- $\mathbb{S}^n$ - the space of symmetric matrices in $\mathbb{R}^{n\times n}$

The scalar product of two elements $x, y \in \mathbb{R}^n$ is denoted by $\langle x, y\rangle$ and the Euclidean norm by $|x|^2 = \sum_{i=1}^n (x_i)^2$. We define I_n to be the n-dimensional identity matrix. The entries of an $(n \times m)$-dimensional matrix A are given by A_{ij} with $1 \leq i \leq n$ and $1 \leq j \leq m$, and we denote its transpose by A^*. For the $(n \times m)$-dimensional matrix A, we define the following matrix norm induced by the Euclidean norm:

$$\|A\| = \max_{|x|=1} |Ax| = \max_{x\neq 0} \frac{|Ax|}{|x|}.$$

The partial derivative with respect to t of a map $v(t, x, y)$ with $v : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is denoted as v_t, its first-order partial derivative with respect to x as $\nabla_x v$ and its first-order partial derivative with respect to y as $\nabla_y v$ (whenever they exist). To denote the i-th first derivative of the function v with respect to x_i, $1 \leq i \leq n$, we write $\nabla_{x_i} v$ (whenever it exists). With $\nabla_{xx} v$, resp. $\nabla_{yy} v$, we denote the second-order partial derivative with respect to x, resp. y. For its second-order partial derivative with respect to x_i and x_j, $1 \leq i, j \leq n$, we write $\nabla_{x_i x_j} v$ (whenever it exists).

We consider the following function spaces.

- $C([0, T] \times \mathbb{R}^m; \mathbb{R}^n)$ - the set of all continuous functions on $[0, T] \times \mathbb{R}^m$ taking values in $\mathbb{R}^n$. If it is not important to specify the range of the function, we write $C([0, T] \times \mathbb{R}^m)$.
- $C_b([0, T] \times \mathbb{R}^m; \mathbb{R}^n)$ - the set of all continuous and bounded functions on $[0, T] \times \mathbb{R}^m$ taking values in $\mathbb{R}^n$. If it is not important to specify the range of the function, we write $C_b([0, T] \times \mathbb{R}^m)$.
- $C^{k,\ell}([0, T] \times \mathbb{R}^m; \mathbb{R}^n)$ - for $k, \ell \in \mathbb{N}$, this is the set of all $\mathbb{R}^n$-valued functions $\varphi(t, x)$, $(t, x) \in [0, T] \times \mathbb{R}^m$ that are k-times continuously differentiable in t and ℓ-times

continuously differentiable in x. If it is not important to specify the range of the function, we write $C^{k,\ell}([0,T]\times\mathbb{R}^m)$.

- $C_b^{k,\ell}([0,T]\times\mathbb{R}^m;\mathbb{R}^n)$ - the set of those $\varphi\in C^{k,\ell}([0,T]\times\mathbb{R}^m;\mathbb{R}^n)$ such that all the partial derivatives are uniformly bounded. If it is not important to specify the range of the function, we write $C_b^{k,\ell}([0,T]\times\mathbb{R}^m)$.
- $C_p^{k,\ell}([0,T]\times\mathbb{R}^m;\mathbb{R}^n)$ - the set of all functions in $C^{k,\ell}([0,T]\times\mathbb{R}^m;\mathbb{R}^n)$ taking values in $\mathbb{R}^n$ which satisfy a polynomial growth condition on $([0,T]\times\mathbb{R}^m)$. If it is not important to specify the range of the function, we write $C_p^{k,\ell}([0,T]\times\mathbb{R}^m)$.
- $C_c([0,T]\times\mathbb{R}^m;\mathbb{R}^n)$ - the set of all continuous functions on $[0,T]\times\mathbb{R}^m$ having compact support.
- $L^p([0,T];\mathbb{R}^n)$ - the set of Lebesgue measurable functions $\varphi:[0,T]\to\mathbb{R}^n$ such that $\int_0^T|\varphi(t)|^p dt<\infty$ $(p\in[1,\infty))$.
- $L^\infty([0,T];\mathbb{R}^n)$ - the set of essentially bounded Lebesgue measurable functions $\varphi:[0,T]\to\mathbb{R}^n$.

Let $T>0$ be a finite time horizon. In the following ν will denote a d-dimensional Brownian stochastic basis, that is $\nu=(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},P,W)$, with

- W - d-dimensional Brownian motion,
- $(\Omega,\mathcal{F},P)$ - probability space,
- $(\mathcal{F}_t)_{t\geq 0}$ - filtration,
- $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0})$ - filtered probability space.

We denote with $\mathcal{B}(\Omega)$ the Borel σ-field generated by all the open sets in $\Omega´$. On $\nu=(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},P,W)$, we introduce the following spaces.
For all $m\in\mathbb{N}$ and any $t\in[0,T]$,

- $L_\nu^p(\Omega;\mathbb{R}^n)$ - the space of all $\mathcal{F}$-measurable $\mathbb{R}^n$-valued random variables X such that $\|X\|_{L^p}=E\left[|X|^p\right]^{\frac{1}{p}}<\infty$. If $n=1$, one chooses often the shorter notation $L_\nu^p(\Omega)\equiv L_\nu^p(\Omega;\mathbb{R})$
- $L_\nu^\infty(\Omega;\mathbb{R}^n)$ - the space of all bounded $\mathcal{F}$-measurable $\mathbb{R}^n$-valued random variables such that $\|X\|_{L^\infty}=\sup_{\omega\in\Omega}|X(\omega)|<\infty$. Again for $n=1$, often the shorter notation $L_\nu^\infty(\Omega)\equiv L_\nu^\infty(\Omega;\mathbb{R}^n)$ is used.
- $L_\nu^p([0,T]\times\Omega;\mathbb{R}^n)$ - the set of all $(\mathcal{F}_t)_{t\geq 0}$-adapted $\mathbb{R}^n$-valued processes $f(t,\omega)$ such that $\|f\|_T^p=E\left[\int_0^T|f(t,\cdot)|^p dt\right]<\infty$. Often we omit the dependence on Ω and write $L_\nu^p([0,T];\mathbb{R}^n)$ instead.

- $L^{p,loc}_{\nu}([0,T]\times\Omega;\mathbb{R}^n)$ - the set of all $(\mathcal{F}_t)_{t\geq 0}$-adapted $\mathbb{R}^n$-valued processes $f(t,\omega)$ such that $\int_0^T |f(t,\cdot)|^p dt < \infty$, P-a.s. Again, we often omit the dependence on Ω and write $L^{p,loc}_{\nu}([0,T];\mathbb{R}^n)$ instead.
- $\mathcal{S}^p_{\nu}([0,T];\mathbb{R}^n)$ - the space of all $(\mathcal{F}_t)_{t\geq 0}$-adapted $\mathbb{R}^n$-valued processes Y such that $\|Y\|_{\mathcal{S}^p} = E\left[\left(\sup_{t\in[0,T]}|Y_t|\right)^p\right]^{\frac{1}{p}} < \infty$.
- $\mathcal{S}^\infty_{\nu}([0,T];\mathbb{R}^n)$ - the space of essentially bounded $(\mathcal{F}_t)_{t\geq 0}$-adapted $\mathbb{R}^n$-valued processes.
- $\mathcal{H}^p_{\nu}([0,T];\mathbb{R}^n)$ - the space of all $(\mathcal{F}_t)_{t\geq 0}$-adapted $\mathbb{R}^n$-valued processes Z such that $\|Z\|_{\mathcal{H}^p} = E\left[\left(\int_0^T |Z_t|^2 ds\right)^p\right]^{\frac{1}{p}} < \infty$.
- $\mathcal{M}^2[0,T]$ - the set of square-integrable martingales
- $\mathcal{M}^2_c[0,T]$ - the set of square-integrable continuous martingales
- $\mathcal{M}^{2,loc}[0,T]$ - the set of locally square integrable martingales
- $\mathcal{M}^{2,loc}_c[0,T]$ - the set of locally square integrable continuous martingales

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Selbständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Berlin, den 24. Juli 2013 Katrin Eichmann, geb. Bernoth